

Upper bounds on Liouville first passage percolation and Watabiki's prediction

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Abstract

Given a planar continuous Gaussian free field h in a domain D with Dirichlet boundary condition and any $\delta > 0$, we let $\{h_\delta(v) : v \in D\}$ be a real-valued smooth Gaussian field where $h_\delta(v)$ is the average of h in a circle of radius δ with center z . For $\gamma > 0$, we study the Liouville first passage percolation (in scale δ), i.e., the shortest path metric in D where the length of each path P is given by $\int_P e^{\gamma h_\delta(z)} |dz|$. We show that the distance between two typical points is $O(\delta^{c^* \gamma^{4/3} / \log \gamma^{-1}})$ for all sufficiently small but fixed $\gamma > 0$ and some constant $c^* > 0$. In addition, we obtain similar upper bounds on the Liouville first passage percolation for discrete Gaussian free fields, as well as the Liouville graph distance which roughly speaking is the minimal number of Euclidean balls with comparable Liouville quantum gravity measure whose union contains a continuous path between two endpoints. Our results contradict with some reasonable interpretations of Watabiki's prediction (1993) on the random metric of Liouville quantum gravity at high temperatures.

Key words and phrases. Liouville quantum gravity (LQG), Gaussian free field (GFF), First passage percolation (FPP).

1 Introduction

Let $D \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary and for $\delta > 0$ let $D_\delta = \{v \in D : d(v, \partial D) > \delta\}$. In this paper we only consider domains D such that $V \equiv [0, 1]^2 \subseteq D_\varepsilon$ for some fixed $\varepsilon > 0$. Let h be a continuous Gaussian free field (GFF) on D with Dirichlet boundary conditions. For an introduction to GFF including various formal constructions, see, e.g., [40, 7]. Although it is not possible to make sense of h as a function on D , it is regular enough so that we can interpret its Lebesgue integrals over sufficiently nice Borel sets in a rigorous way. In particular we can take its average along a circle of radius δ around v (where $d_{\ell_2}(v, \partial D) > \delta$) and define the *circle average process* $\{h_\delta(v) : v \in D, d_{\ell_2}(v, \partial D) > \delta\}$ which is a centered Gaussian field with covariance

$$\text{Cov}(h_\delta(v), h_{\delta'}(v')) = \int_{\partial B_\delta(v) \times \partial B_{\delta'}(v')} G_D(z, z') \mu_\delta^v(dz) \mu_{\delta'}^{v'}(dz').$$

Here $B_r(z)$ is the open ball with radius r centered at z , μ_r^z is the uniform probability measure on $\partial B_r(z)$ and $G_D(z, z')$ is the Green function for domain D , which we define by

$$G_D(z, z') = \int_{(0, \infty)} p_D(s; z, z') ds,$$

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where $p_D(s; z, z')$ is the transition probability density of Brownian motion killed when exiting D . It was shown in [21] that there exists a version of the circle average process which is jointly Hölder continuous in v and δ of order $\vartheta < 1/2$ on all compact subsets of $\{(v, \delta) : v \in D, 0 < \delta < d_{\ell_2}(z, \partial D)\}$. Given such an instance of h_δ and a fixed inverse-temperature parameter $\gamma > 0$, we define the *Liouville first-passage percolation (Liouville FPP) metric* $D_{\gamma, \delta}(\cdot, \cdot)$ on V by

$$D_{\gamma, \delta}(v, w) = \inf_P \int_P e^{\gamma h_\delta(z)} |dz|, \quad (1.1)$$

where P ranges over all piecewise C^1 paths in V connecting v and w . The infimum is well-defined and measurable since we are dealing with a continuous field on a compact space. In fact $D_{\gamma, \delta}(\cdot, \cdot)$ does not change if we restrict only to C^1 paths.

Theorem 1.1. *There exists $C_{\gamma, D, \varepsilon} > 0$ (depending on (γ, ε, D)) and positive (small) absolute constants c^*, γ_0 such that for all $\gamma \leq \gamma_0$, we have*

$$\max_{v, w \in V} \mathbb{E} D_{\gamma, \delta}(v, w) \leq C_{\gamma, D, \varepsilon} \delta^{c^* \frac{\gamma^{4/3}}{\log \gamma^{-1}}}.$$

Another related notion of random metric comes from the *Liouville quantum gravity* (LQG) measure M_γ^D on D . For any $\gamma < 2$, M_γ^D is defined as the almost sure weak limit of the sequence of measures $M_{\gamma, n}^D$ given by

$$M_{\gamma, n}^D = e^{\gamma h_{2^{-n}}(z)} 2^{-\pi n \gamma^2 / 2} \sigma(dz), \quad (1.2)$$

where σ is the Lebesgue measure. Much on the LQG measure has been understood (see e.g., [27, 21, 37, 38, 39]) including the existence of the limit in (1.2), the uniqueness in law for the limiting measure via different approximation schemes, as well as a KPZ correspondence through a uniformization of the random lattice seen as a Riemann surface. Our focus in the present article is the *metric* aspect of the LQG. Given $\delta \in (0, 1)$, we say a closed Euclidean ball $B \subseteq D$ is a (M_γ^D, δ) -ball if $M_\gamma^D(B) \leq \delta^2$ and the center of B is rational (to avoid unnecessary measurability consideration). We then define the *Liouville graph distance* $\tilde{D}_{\gamma, \delta}(v, w)$ between $v, w \in V$ as the minimum number of (M_γ^D, δ) balls whose union contains a path between v and w . We name this metric as Liouville graph distance since it corresponds to the shortest path distance on a graph indexed on \mathbb{Q}^2 where neighboring relation corresponds to the intersection of the (M_γ^D, δ) balls. A very related graph distance was mentioned in [33] which proposed to keep dividing each squares until the LQG measure is below δ . We chose our notion of Liouville graph distance for the reason that it seems to have more desirable invariant properties, though we expect our bound (as well as our proof) to extend to the other notion too.

Theorem 1.2. *There exists $C_{\gamma, D, \varepsilon} > 0$ (depending on (γ, ε, D)) and positive (small) absolute constants c^*, γ_0 such that for all $\gamma \leq \gamma_0$, we have*

$$\max_{v, w \in V} \mathbb{E} \tilde{D}_{\gamma, \delta}(v, w) \leq C_{\gamma, D, \varepsilon} \delta^{-1 + c^* \frac{\gamma^{4/3}}{\log \gamma^{-1}}}.$$

Finally, it is as natural to consider Liouville FPP for discrete GFF (which was explicitly mentioned in [5]). Given a two-dimensional box V_N of side length N , the discrete GFF $\{\eta_{N, v} : v \in V_N\}$ with Dirichlet boundary conditions is a mean-zero Gaussian process such that

$$\eta_{N, v} = 0 \text{ for all } v \in \partial V_N, \text{ and } \mathbb{E} \eta_{N, v} \eta_{N, w} = G_{V_N}(v, w) \text{ for all } v, w \in V_N,$$

where $G_{V_N}(v, w)$ is the Green's function of simple random walk on V_N . As before, for a fixed inverse-temperature parameter $\gamma > 0$, we define the Liouville FPP metric $D_{\gamma, N}(\cdot, \cdot)$ on V_N by

$$D_{\gamma, N}(v_1, v_2) = \min_{\pi} \sum_{v \in \pi} e^{\gamma \eta_{N, v}}, \quad (1.3)$$

where π ranges over all paths in V_N connecting v_1 and v_2 . As we explain in Section 7, the proof of Theorem 1.1 can be adapted to derive the following result.

Theorem 1.3. *Given any fixed $0 < \varepsilon < 1/2$, there exists $C_{\gamma, \varepsilon} > 0$ (depending on (γ, ε)) and positive (small) absolute constant c^*, γ_0 such that for all $\gamma \leq \gamma_0$, we have*

$$\max_{v, w \in V_{N, \varepsilon}} \mathbb{E} D_{\gamma, N}(v, w) \leq C_{\gamma} N^{1 - c^* \frac{\gamma^{4/3}}{\log \gamma^{-1}}},$$

where $V_{N, \varepsilon}$ is the square $\{v \in V_N : d_{\infty}(v, \partial V_N) \geq \varepsilon N\}$.

Remark 1.4. Theorem 1.3 still holds if we restrict π to be a path within $V_{N, \varepsilon}$ in (1.3).

1.1 Discussion on Watabiki's prediction

The Liouville FPP and the Liouville graph distance are two (related) natural discrete approximations for the random metric associated with the Liouville quantum gravity (LQG) [35, 21, 38]. Precise predictions on various exponents regarding to LQG metric have been made by Watabiki [41] (see also, [3, 2]). In particular, the Hausdorff dimension for the LQG metric is predicted to be

$$d_H(\gamma) = 1 + \frac{\gamma^2}{4} + \sqrt{\left(1 + \frac{\gamma^2}{4}\right)^2 + \gamma^2}. \quad (1.4)$$

The prediction in (1.4) was widely believed. In a recent work [33], Miller and Sheffield introduced and studied a process called *quantum Loewner evolution*. As a *byproduct* of their work they gave a non-rigorous analysis on exponents of the LQG metric which matched Watabiki's prediction — we also note that in [33] the authors did express some reservations on their non-rigorous analysis. For other discussions on Watabiki's prediction in mathematical literature, see e.g., [32, 25].

The precise mathematical interpretation of Watabiki's prediction is not completely clear to us. However, there are a number of reasonable “folklore” interpretations that seem to be widely accepted. For the Liouville graph distance, the scaling exponent $\chi = -\lim_{\delta \rightarrow 0} \frac{\mathbb{E} \tilde{D}_{\gamma, \delta}(v, w)}{\log \delta}$ is expected to exist and is expected to be given by (here we take v, w as two fixed generic points in the domain)

$$\chi = \frac{2}{d_H(\gamma)} = 1 - O_{\gamma \rightarrow 0}(\gamma^2), \quad (1.5)$$

where in the last step we plugged in (1.4). A similar interpretation to (1.5) appeared in [25, Conjecture 1.14] though the graph structure considered in [25] is based on the peanosphere construction of LQG and so far we see no mathematical connection to Liouville graph distance considered in the present article. Note that there is a difference of factor of 2, which is due to the fact that for the graph defined in [25] on average each ball contains LQG measure about ε (in their notation) while in our construction each ball contains LQG measure δ^2 . We see that Theorem 1.2 contradicts (1.5).

There are also reasonable interpretations of Watabiki's prediction for Liouville FPP.¹ For instance, see [2, Equation (17), (18)]. We would like to point out that in [2, Equation (17)] the term ρ_{δ} was not

¹For instance, we learned from Rémi Rhodes and Vincent Vargas that, according to [41], the physically appropriate approximation for the γ -LQG metric should involve $\inf_P \int_P e^{\frac{\gamma}{d_H(\gamma)} h_{\delta}(z)} |dz|$, i.e., the parameter in the exponential of GFF is $\gamma/d_H(\gamma)$ instead of γ .

defined — some reasonable interpretations include $\rho_\delta = e^{\gamma h_\delta(z)}$ and $\rho_\delta = e^{\gamma h_\delta(z)} \delta^{\frac{\gamma^2}{2}}$ as well as possibly replacing γ by $\frac{\gamma}{d_H(\gamma)}$ as suggested in the footnote. For all these interpretations, [2, Equation (18)] would then imply that there exist constants $c, C > 0$ such that for sufficiently small but fixed $\gamma > 0$ the Liouville FPP distance between two generic points is between $\delta^{C\gamma^2}$ and $\delta^{c\gamma^2}$ as $\delta \rightarrow 0$. However, Theorem 1.1 contradicts with all aforementioned interpretations of (1.4) for Liouville FPP at high temperatures.

Currently, we do not have any reasonable conjecture on the precise value of the exponent for either Liouville FPP or Liouville graph distance — we regard a precise computation of the exponent for either of the two metrics as a major challenge.

1.2 Discussion on non-universality

Combined with [19], Theorem 1.3 shows that the weight exponent for first passage percolation on the exponential of log-correlated Gaussian fields is non-universal, i.e., the exponents may differ for different families of log-correlated Gaussian fields. In contrast, we note that the behavior for the maximum is universal among log-correlated Gaussian fields (see e.g., [11, 10, 31, 17]) in a sense that their expectations are the same up to additive $O(1)$ term and that the laws of the centered maxima for all these fields are in the same universal family known as Gumbel distribution with random shifts (but the random shifts may not have the same law for different fields).

While non-universality suggests subtlety for the weight exponent of Liouville FPP, the proof in the present article does not see complication due to such subtlety. In fact, our proof should be adaptable to general log-correlated Gaussian fields with \star -scale invariant kernels as in [20]. The following question remains an interesting challenge, especially (in light of the non-universality) for log-correlated Gaussian fields for which a kernel representation is not known to exist.

Question 1.5. *Let $\{\varphi_{N,v} : v \in V_N\}$ be an arbitrary mean-zero Gaussian field satisfying $|\mathbb{E}\varphi_{N,v}\varphi_{N,u} - \log \frac{N}{1+\|u-v\|}| \leq K$. Does an analogue of Theorem 1.3 hold for C_γ, c^* depending on K ?*

1.3 Further related works

Much effort has been devoted to understanding classical first-passage percolation (FPP), with independent and identically distributed edge/vertex weights. We refer the reader to [4, 24] and their references for reviews of the literature on this subject. We argue that FPP with strongly-correlated weights is also a rich and interesting subject, involving questions both analogous to and divergent from those asked in the classical case. Since the Gaussian free field is in some sense the canonical strongly-correlated random medium, we see strong motivation to study Liouville FPP.

More specifically, as mentioned earlier Liouville FPP and Liouville graph distance play key roles in understanding the random metric associated with the Liouville quantum gravity (LQG). We remark that the random metric of LQG is a major open problem, even just to make rigorous sense of it (we refer to [36] for a rather up-to-date review). In a recent series of works of Miller and Sheffield, much understanding has been obtained for the LQG metric (in the special case when $\gamma = \sqrt{8/3}$), and we note that an essentially equivalent metric to Liouville graph distance was mentioned in [33] as a natural approximation. While no mathematical result was obtained (perhaps not attempted either) on such approximations, the main achievement of this series of works by Miller and Sheffield (see [33, 34] and references therein) is to produce candidate scaling limits and to establish a deep connection to the Brownian map. Our approach is different, in the sense that we aim to understand the random metric of LQG via approximations by natural discrete metrics. We also note that in a recent work [25] some upper and lower bounds have

been obtained for a type of distance related to LQG and that their bounds are consistent with Watabiki's prediction. We further remark that currently we see no connection between our work and [33, 34, 25].

Furthermore, we expect that Liouville FPP metric and Liouville graph distance are related to the heat kernel estimate for Liouville Brownian motion (LBM) — in fact, we expect a direct and strong connection between Liouville graph distance and the LBM heat kernel. The mathematical construction (of the diffusion) for LBM was provided in [22, 6] and the heat kernel was constructed in [23]. The LBM is closely related to the geometry of LQG; in [12, 8] the KPZ formula was derived from Liouville heat kernel. In [32] some nontrivial bounds for LBM heat kernel were established. A very interesting direction is to compute the heat kernel of LBM with high precision.

There has been a number of other recent works on Liouville FPP (while they focus on the case for the discrete GFF, these results are expected to extend to the case of continuous GFF). In a recent work [13], it was shown that at high temperatures the appropriately normalized Liouville FPP converges subsequentially in the Gromov-Hausdorff sense to a random metric on the unit square, where all the (conjecturally unique) limiting metrics are homeomorphic to the Euclidean metric. We remark that the proof method in the current paper bears little similarity to that in [13]. In a very recent work [18], it was shown that the dimension of the geodesic for Liouville FPP is strictly larger than 1. In fact, in [18] it proved that all paths with dimension close to 1 has weight exponent close to 1, which combined with Theorem 1.3 yields that the lower bound on the dimension of the geodesic. While both the proofs in [18] and the present article use multi-scale analysis method, the details are drastically different.

Finally, we would like to mention that there has been a few other recent works on the metric properties of two-dimensional discrete GFF, including [30] on the random pseudo-metric on a graph defined via the zero-set of the Gaussian free field on its metric graph, [16] on the chemical distance on the level sets of GFF, and [9] on the effective resistance metric on the random network where each edge (u, v) is assigned a resistance $e^{\gamma(\eta_u + \eta_v)}$. In particular, the work of [9] implies that the typical effective resistance between two vertices of Euclidean distance N behaves like $N^{o(1)}$. Combined with Theorem 1.3, this implies that (somewhat mysteriously) perturbing \mathbb{Z}^2 by assigning weights which are exponentials of GFF drastically distorts the shortest path distance but more or less preserves the effective resistance of \mathbb{Z}^2 .

1.4 A historical remark and the proof strategy

Our proof strategy naturally inherits that of [15] which proved a weak version of Theorem 1.3 in the context of BRW, and we encourage the reader to flip through [15] (in particular Section 1.2) which contains a prototype of the multi-scale analysis carried out in the current paper. In fact, prior to the present article, we posted an article [14] on arXiv which proved that the weight exponent is less than $1 - \gamma^2/10^3$. Our present article proves a stronger result than [14]. In addition, the proof simplifies that of [14] and is self-contained. As a result, [14] will be superseded by the present article and will not be published anywhere.

However, some historical remarks might be interesting and helpful. During the work of [14], we had in mind that the second leading term for the weight exponent is of order γ^2 in light of (1.4). As a result, we followed [15] and designed a strategy of constructing light crossings inductively to prove an upper bound of $1 - \gamma^2/10^3$. In the multi-scale construction, the order of γ^2 is exactly the order of both the gain and the loss for our strategy, and thus a much delicate analysis was carried out in [14] since we fought between two constants for the loss and the gain. A curious reader may quickly flip through [14] for an impression on the level of technicality.

A key component in both [15, 14] is an inductive construction where we construct light crossings in a bigger scale from crossings in smaller scale, where we will switch between two layers of candidate crossings in smaller scale the value of Gaussian variables in the bigger scale (note that there is a hierarchical

structure for both BRW and GFF). In those papers, we used vertical crossings as our switching gadgets to connect horizontal crossings in top and bottom layers. A crucial improvement in the current article arises from a simple observation that a *sloped* switching gadget is much more efficient (see Figure 5). In order to give a flavor of how it works we discuss the following toy problem.

Let $\Gamma = \Gamma(\gamma)$ be a large positive number and $\{\zeta(v) : v \in V^\Gamma\}$ be a continuous, centered Gaussian field on the rectangle $V^\Gamma = [0, \Gamma] \times [0, 1]$. Suppose that the ζ satisfies the following properties:

- (a) $\text{Var}(\zeta(v)) = 1$ for all $v \in V^\Gamma$.
- (b) For any straight line segment \mathcal{L} , $\text{Var}(\int_{\mathcal{L}} \zeta(z)|dz|) = O(|\mathcal{L}|)$, where $|\mathcal{L}|$ is the (euclidean) length of \mathcal{L} . Furthermore if $v \in \mathbb{R}^2$ is orthogonal to \mathcal{L} such that $\|v\| = \Omega(1)$, then

$$\text{Var}\left(\int_{\mathcal{L}} \zeta(z)|dz| - \int_{\mathcal{L}+v} \zeta(z)|dz|\right) = \Theta(|\mathcal{L}|).$$

We want to construct a piecewise smooth path P connecting the shorter boundaries of V^Γ that has a small “random length” given by $\int_P e^{\gamma\zeta(z)}|dz|$. Due to condition (a), we can approximate $e^{\gamma\zeta(z)}$ with $1 + \gamma\zeta(z) + \frac{\gamma^2}{2}$ when γ is sufficiently small. Thus the random length of P is approximately

$$(1 + \frac{\gamma^2}{2})|P| + \gamma \int_P \zeta(z)|dz|. \quad (1.6)$$

Henceforth we will treat the above expression as the “true” random length of P . Now consider the segments $\mathcal{L}_1 = [0, \Gamma] \times \{0.75\}$ and $\mathcal{L}_2 = [0, \Gamma] \times \{0.25\}$. Choose β such that $\Gamma \gg \beta \gg 1$ and divide \mathcal{L}_i (here $i \in [2]$) into segments $\mathcal{L}_{i,1}, \mathcal{L}_{i,2}, \dots, \mathcal{L}_{i,\Gamma/\beta}$ of length β from left to right. Given $i_j \in \{1, 2\}$ for each $j \in [\Gamma/\beta]$ (called a *strategy*), we can construct a crossing i.e. a path connecting the left and right boundaries of V^Γ as follows. If $i_j = i_{j+1}$, let \mathcal{L}'_j be the segment $\mathcal{L}_{i_j,j}$. Otherwise set \mathcal{L}'_j as the segment joining the left endpoints of $\mathcal{L}_{i_j,j}$ and $\mathcal{L}_{i_{j+1},j+1}$ (the sloped gadget). It is clear that the segments $\mathcal{L}'_1, \mathcal{L}'_2, \dots, \mathcal{L}'_{\Gamma/\beta}$ define a crossing. The random length (see (1.6)) of this crossing is given by

$$(1 + \frac{\gamma^2}{2})\Gamma + (1 + \frac{\gamma^2}{2}) \sum_{j \in [\Gamma/\beta-1]} \mathbf{1}_{\{i_j \neq i_{j+1}\}} (|\mathcal{L}'_j| - \beta) + \gamma \sum_{j \in [\Gamma/\beta]} \int_{\mathcal{L}'_j} \zeta(z)|dz|.$$

Since $\beta \gg 1$, $|\mathcal{L}'_j| - \beta = \sqrt{O(1) + \beta^2} - \beta = O(\beta^{-1})$ whenever $i_j \neq i_{j+1}$. On the other hand by condition (b), $(\int_{\mathcal{L}'_j} \zeta(z)|dz| - \int_{\mathcal{L}_{i_j,j}} \zeta(z)|dz|)$ is a centered Gaussian variable with variance

$$\text{Var}\left(\int_{\mathcal{L}'_j} \zeta(z)|dz| - \int_{\mathcal{L}_{i_j,j}} \zeta(z)|dz|\right) = O(|\mathcal{L}'_j|) = O(\beta),$$

whenever $i_j \neq i_{j+1}$ and thus

$$\mathbb{E}\left(\gamma \int_{\mathcal{L}'_j} \zeta(z)|dz| - \gamma \int_{\mathcal{L}_{i_j,j}} \zeta(z)|dz|\right)^+ = O(\sqrt{\beta}).$$

Therefore if we choose our strategy so that $i_j \neq i_{j+1}$ only on a *fixed* set $J = \{j_1, j_2, \dots, j_{|J|}\}$, then we can bound (from above) the expected random length of the crossing by

$$(1 + \frac{\gamma^2}{2})\Gamma + (1 + \frac{\gamma^2}{2})|J|C\beta^{-1} + \gamma|J|C'\sqrt{\beta} + \gamma \sum_{k \in [J]} \mathbb{E}\left(\int_{\mathcal{L}_{i_k,j_k}} \zeta(z)|dz|\right). \quad (1.7)$$

Here C, C' are positive constants and $\overline{\mathcal{Z}}_{i_{j_k}, J}$ is the union of segments $\mathcal{L}_{i_{j_{k-1}}, j_{k-1}+1}, \mathcal{L}_{i_{j_{k-1}}, j_{k-1}+2}, \dots, \mathcal{L}_{i_{j_{k-1}}, j_k}$ with $j_0 = 0$. Now notice that

$$\mathbb{E} \left(\int_{\overline{\mathcal{Z}}_{i_{j_k}, J}} \zeta(z) |dz| \right) = \frac{1}{2} \mathbb{E} (-1)^{i_{j_k}+1} \left(\int_{\mathcal{Z}_{1,J}} \zeta(z) |dz| - \int_{\mathcal{Z}_{2,J}} \zeta(z) |dz| \right),$$

as $\int_{\mathcal{Z}_{1,J}} \zeta(z) |dz|$ and $\int_{\mathcal{Z}_{2,J}} \zeta(z) |dz|$ are centered. But by condition (b), $\int_{\mathcal{Z}_{1,J}} \zeta(z) |dz| - \int_{\mathcal{Z}_{2,J}} \zeta(z) |dz|$ is a centered Gaussian variable with variance $\geq c(j_k - j_{k-1})\beta$ for some positive constant c . Thus if we allow $j_k - j_{k-1}$ to be only large enough so that

$$\frac{\gamma}{2} \mathbb{E} \left| \int_{\mathcal{Z}_{1,J}} \zeta(z) |dz| - \int_{\mathcal{Z}_{2,J}} \zeta(z) |dz| \right| \geq 2 \left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta},$$

and set $i_j = 1$ or 2 accordingly as $\int_{\mathcal{Z}_{1,J}} \zeta(z) |dz| - \int_{\mathcal{Z}_{2,J}} \zeta(z) |dz| < \text{or} > 0$, then

$$\gamma \mathbb{E} \left(\int_{\overline{\mathcal{Z}}_{i_{j_k}, J}} \zeta(z) |dz| \right) \leq -2 \left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right),$$

and

$$|J| = \frac{\Omega(\frac{\Gamma}{\beta})}{\frac{\left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right)^2}{\beta \gamma^2}} = \frac{\Omega(\Gamma \gamma^2)}{\left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right)^2}.$$

Let us call the path given by this strategy as P^* . Plugging the previous two expressions into (1.7), we find that the expected random length of P^* can be at most

$$\begin{aligned} & \left(1 + \frac{\gamma^2}{2} \right) \Gamma + |J| \left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right) - 2|J| \left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right) \\ &= \left(1 + \frac{\gamma^2}{2} \right) \Gamma - \frac{\Omega(\Gamma \gamma^2)}{\left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right)^2} \left(\left(1 + \frac{\gamma^2}{2} \right) C \beta^{-1} + 2\gamma C' \sqrt{\beta} \right) \\ &= \left(1 + \frac{\gamma^2}{2} \right) \Gamma - \frac{\Omega(\Gamma \gamma^2)}{\gamma \sqrt{\beta} + \beta^{-1}}. \end{aligned}$$

The above expression is minimized for $\beta = \Theta(\gamma^{-2/3})$ and the optimal value is $\Gamma(1 - \Omega(\gamma^{4/3}))$ when γ is small. This shows, on a high level, why we get a contraction as in Theorem 1.1.

We remark that the simple observation on the slopped switching strategy is more natural when considering continuous path in the plane — this is why our main proof focuses on the case of continuous GFF. In the case for discrete GFF, we first bound the distance minimizing the lengths over all continuous path and then argue that for each continuous path there is a lattice path whose weight grows by a factor that is negligible.

We now give a brief guide on the organization. In Section 2, we introduce a new Gaussian field which has a simpler hierarchical structure than the circle average process — our main proof will be carried out for this new field. In Section 3 we describe our inductive construction on light crossings as scales increase and in particular we introduce the aforementioned sloped switching gadget. In Section 4 we analyze the construction in Section 3 and derive an upper bound on the weight exponent for the Gaussian field from Section 2. In Section 5, we show that the circle average process of GFF is well-approximated by the field from Section 2, thereby proving Theorem 1.1. In Section 6, we prove Theorem 1.2 by relating the Liouville graph distance to Liouville FPP and applying Theorem 1.1. Finally, in Section 7 we explain how to adapt the proof to deduce Theorem 1.3.

1.5 Conventions, notations and some useful definitions

We assume that γ is small enough (less than some small, positive absolute constant) for our bounds or inequalities to hold although we keep this implicit in our discussions. Γ is the smallest (integral) power of 2 that is $\geq \gamma^{-2}$. Thus $1 \leq \Gamma\gamma^2 < 2$. (It will be clear from our analysis that any exponent $< -4/3$ should work.) For any $w \in \mathbb{R}^2$, $\ell \in \mathbb{N}$ and $r > 0$, $V_\ell^{r,w}$ denotes the rectangle $w + [0, r2^{-\ell}] \times [0, 2^{-\ell}]$. We will suppress ℓ or w from this notation whenever they are 0. We will also omit r when it is 1. We call two rectangles R and R' to be *copies* of each other if R can be obtained from R' via translation and / or rotation by an angle. The rectangles R and R' are called *non-overlapping* if their interiors are disjoint. If R and R' have same dimensions then we say that they are *adjacent* if they share one of their shorter boundary segments. A *smooth path* is a C^1 map $P : [0, 1] \rightarrow \mathbb{R}^2$. We also use P to denote the image set of P which is a subset of \mathbb{R}^2 . This distinction should be clear from the context. For any rectangle $R = [a, b] \times [c, d]$ with sides parallel to the coordinate axes, we define its left, right, upper and lower boundary segments in the obvious way and denote them as $\partial_{\text{left}}R$, $\partial_{\text{right}}R$, $\partial_{\text{up}}R$ and $\partial_{\text{down}}R$ respectively. Thus $\partial_{\text{left}}R$ is the path described by $(a, c + t(d - c)); t \in [0, 1]$ etc. For convenience, we will identify (and denote) the points in \mathbb{R}^2 as complex numbers. The euclidean distance $d_{\ell_2}(S, S')$ between any two subsets S and S' of \mathbb{R}^2 is defined as $\inf_{v \in S, v' \in S'} |v - v'|$.

For (nonnegative) functions $F(\cdot)$ and $G(\cdot)$ we write $F = O(G)$ (or $\Omega(G)$) if there exists an absolute constant $C > 0$ such that $F \leq CG$ (respectively $\geq CG$) everywhere in the domain. If the constant depends on variables x_1, x_2, \dots, x_n , we modify these notations as $O_{x_1, x_2, \dots, x_n}(G)$ and $\Omega_{x_1, x_2, \dots, x_n}(G)$ respectively. $F = \Theta(G)$ if F is both $O(G)$ and $\Omega(G)$. For any positive integer i , the notations C_i and c_i indicate positive, absolute constants whose values are assumed to be same throughout the paper. Similarly we use $C_{x_1, x_2, \dots, x_k}, C'_{x_1, x_2, \dots, x_k}, C''_{x_1, x_2, \dots, x_k}, \dots$ to denote fixed positive functions C, C', C'', \dots of x_1, x_2, \dots, x_k . However we keep these qualifications i.e. “positive”, “absolute constant”, “depends on x_1, x_2, \dots, x_k ” etc. implicit in our discussion.

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2 Preliminaries

2.1 White noise decomposition of some Gaussian processes

A white noise W distributed on $\mathbb{R}^2 \times \mathbb{R}^+$ refers to a centered Gaussian process $\{(W, f) : f \in L^2(\mathbb{R}^2 \times \mathbb{R}^+)\}$ whose covariance kernel is given by $\mathbb{E}(W, f)(W, g) = \int_{\mathbb{R}^2 \times \mathbb{R}^+} fg dw ds$. An alternative notation for (W, f) is $\int_{\mathbb{R}^2 \times \mathbb{R}^+} fW(dw, ds)$, which we will use in this paper. For any $D \in \mathcal{B}(\mathbb{R}^2)$ and $I \in \mathcal{B}(\mathbb{R}^+)$, we let $\int_{D \times I} fW(dw, ds)$ denote the variable $\int_{\mathbb{R}^2 \times \mathbb{R}^+} f_{D \times I} W(dw, ds)$, where $f_{D \times I}$ is the restriction of f to $D \times I$. Now define a Gaussian process $\{h'_\delta(v) : v \in D_\delta\}$ as follows:

$$h'_\delta(v) = \int_{D \times (0, \infty)} \left(\int_{\partial B_\delta(v)} p_D(s/2; v', w) \mu_\delta^v(dv') \right) W(dw, ds) \quad (2.1)$$

Since $G_D(v, w) = \int_{(0, \infty)} p_D(s; v, w) ds$, it is easy to check that the processes h_δ and $h_{\delta'}$ are identically distributed for all $\delta \in (0, \text{diam}(D))$. This provides an automatic coupling between h_δ and a “convenient” field (to be defined shortly) which will be useful in our proof of Theorem 1.1 in Section 5. Henceforth we will work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a white noise is defined.

It turns out that the field $\{h_\delta\}_{\delta>0}$ can be reasonably approximated (see Section 5) by a new family of fields which enjoy certain nice properties. To this end, we define a Gaussian process $\eta \equiv \{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2, 0 < \delta < \delta' \leq 1\}$ as:

$$\eta_\delta^{\delta'}(v) = \int_{\mathbb{R}^2 \times [\delta^2, \delta'^2]} p(s/2; v, w) W(dw, ds). \quad (2.2)$$

where $p(s; v, w)$ is the transition probability density function of standard two-dimensional Brownian motion. We can immediately deduce the following properties of η from this representation:

- (a) *Invariance with respect to symmetries of the plane.* Law of η remains same under any distance preserving transformation (i.e. translation, rotation, reflection etc.) of \mathbb{R}^2 .
- (b) *Scaling property.* The fields $\{\eta_{\delta^* \delta}^{\delta^* \delta'}(\delta^* v) : v \in \mathbb{R}^2\}$ and $\{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2\}$ are identically distributed for all $0 < \delta < \delta' \leq 1$ and $\delta^* \in (0, 1]$.
- (c) *Independent increment.* The fields $\{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2\}$ and $\{\eta_{\delta''}^{\delta'''}(v) : v \in \mathbb{R}^2\}$ are independent for all $0 < \delta < \delta' < \delta'' < \delta''' \leq 1$.

We will suppress the superscript δ' in $\eta_\delta^{\delta'}$ whenever $\delta' = 1$. Notice that

$$\text{Var}(\eta_\delta(v)) = \int_{[\delta^2, 1]} p(s; v, v) ds = \pi^{-1} \log \delta^{-1}. \quad (2.3)$$

In Lemma 2.1 we show that $\text{Var}(\eta_\delta(v) - \eta_\delta(w)) = O(\frac{|v-w|^2}{\delta^2})$. As η_δ is a Gaussian process, this property implies by Kolmogorov-Centsov theorem that there is a version of η_δ with continuous sample paths. Thus we can work with a continuous version η_δ for any given δ and hence for any fixed, finite collection of δ 's that we consider at any given instant.

2.2 Some variance and covariance estimates

Lemma 2.1. *For all $v, w \in \mathbb{R}^2$, we have $\text{Var}(\eta_\delta(v) - \eta_\delta(w)) \leq \frac{|v-w|^2}{\delta^2}$ and $\text{Cov}(\eta_{0.5}(v), \eta_{0.5}(w)) = O(1)e^{-\Omega(|v-w|^2)}$.*

Proof. These follow from (2.2) by straightforward computations:

$$\text{Var}(\eta_\delta(v) - \eta_\delta(w)) = \pi^{-1} \int_{[\delta^2, 1]} \frac{1 - e^{-\frac{|v-w|^2}{2s}}}{s} ds \leq \pi^{-1} \int_{s \in [\delta^2, 1]} \frac{|v-w|^2}{2s^2} ds \leq \frac{|v-w|^2}{\delta^2}.$$

Also

$$\text{Cov}(\eta_{0.5}(v), \eta_{0.5}(w)) = (2\pi)^{-1} \int_{[0.25, 1]} \frac{e^{-\frac{|v-w|^2}{2s}}}{s} ds = O(1)e^{-\frac{|v-w|^2}{2}}. \quad \square$$

We need similar results for a different class of random variables as well. To this end we first define some new objects. Let \mathcal{P} be a finite, non-empty collection of smooth paths in \mathbb{R}^2 . A *random polypath* (or simply a polypath) ξ from \mathcal{P} is a collection of $\{0, 1\}$ -valued random variables $\{e_{\xi, P}\}_{P \in \mathcal{P}}$ such that $\bigcup_{P \in \mathcal{P}: e_{\xi, P}=1} P$ is a connected subset of \mathbb{R}^2 . Thus one can view ξ as a random sub-collection of \mathcal{P} forming a connected set. We will often omit the reference to \mathcal{P} when it is clear from the context and simply say that ξ is a polypath. If $X = \{X(v) : v \in D\}$ is a continuous field and ξ is a polypath from \mathcal{P} , then we

define its *weight computed with respect to X* or alternatively *weight computed with X as the underlying field* as the quantity $\sum_{P \in \mathcal{P}} e_{\xi, P} \int_P e^{\gamma X(z)} |dz|$. For continuous random fields $X = \{X(v) : v \in \mathbb{R}^2\}$ and $Y = \{Y(v) : v \in \mathbb{R}^2\}$, and a polypath ξ such that (ξ, X) is independent with Y , consider the random variable

$$Z(\xi, X, Y; \gamma) = \sum_{P \in \mathcal{P}} e_{\xi, P} \int_P e^{\gamma X(z)} Y(z) |dz|. \quad (2.4)$$

It is a simple consequence of Fubini's theorem that $\mathbb{E} Z_{\xi, X, Y; \gamma}$ is finite whenever $\sup_{P \in \mathcal{P}} \mathbb{E} \int_P e^{\gamma X(z)} |dz|$ and $\sup_{w \in \mathbb{R}^2} \mathbb{E} |Y(w)|$ are both finite. In this case we can express $\bar{Z}(\xi, X, Y; \gamma) = \mathbb{E}(Z(\xi, X, Y; \gamma) | Y)$ as

$$\bar{Z}(\xi, X, Y; \gamma) = \sum_{P \in \mathcal{P}} \int_P \mathbb{E}(e_{\xi, P} e^{\gamma X(z)} Y(z) | dz|. \quad (2.5)$$

Furthermore, $\bar{Z}(\xi, X, Y; \gamma)$ is a centered Gaussian variable if Y is a centered Gaussian field. We will omit X and γ in this notation if $X \equiv 0$. Another quantity of interest is the expected weight of ξ computed with respect to X i.e.

$$L(\xi, X; \gamma) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{P}} \int_P \mathbb{E}(e_{\xi, P} e^{\gamma X(z)} | dz|. \quad (2.6)$$

Now fix some $N \in \mathbb{N}$, $w \in \mathbb{R}^2$ and $v \in \mathbb{R}$. For $v \in \{w, w + \iota v\}$ (here $\iota = \sqrt{-1}$) subdivide the rectangle $V_{2m\Gamma}^{N\Gamma; v}$ into N non-overlapping translates of $V_{2m\Gamma}^\Gamma$ namely $R_{1,v}, R_{2,v}, \dots, R_{N,v}$ ordered from left to right. Suppose that $\xi_{j,v}$ is a polypath contained in $R_{j,v}$ such that $(X, \xi_{j,v})$ is independent with $\eta_{0.5}$ and $L(\xi_{j,v}, X; \gamma) = 1$. Our next lemma deals with the random variables $\sum_{j \in [N]} \bar{Z}(X, \xi_{j,v}, \eta_{0.5}; \gamma)$.

Lemma 2.2. *If $N\Gamma^{-1} \geq 1$ and $v \geq 0.1$, then there exist c_1 and C_1 such that*

$$\sqrt{\text{Var}\left(\sum_{j \in [N]} \bar{Z}(X, \xi_{j, w + \iota v}, \eta_{0.5}; \gamma) - \sum_{j \in [N]} \bar{Z}(X, \xi_{j, w}, \eta_{0.5}; \gamma)\right)} \geq c_1 \sqrt{N\Gamma} - C_1 N\Gamma^{-1}.$$

On the other hand $\text{Var}\left(\sum_{j \in [N]} \bar{Z}(X, \xi_{j, w}, \eta_{0.5}; \gamma)\right) = O(N\Gamma + N^2\Gamma^{-2})$ for all N .

Proof. First we will show that $\text{Var}(\bar{Z}(X, \xi_{j,v}, \eta_{0.5}; \gamma) - \bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5}))$ is small. To this end let $u \in R_{j,v}$. By Fubini, we can write $\text{Var}(\bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5}) - \eta_{0.5}(u))$ as

$$\int_{[0,1]^2} \text{Cov}\left(\eta_{0.5}(v_j + \Gamma^{-1}s) - \eta_{0.5}(u), \eta_{0.5}(v_j + \Gamma^{-1}t) - \eta_{0.5}(u)\right) ds dt,$$

where v_j is the upper-left vertex of the rectangle $R_{j,v}$. Since the diameter of $R_{j,v}$ is $O(\Gamma^{-1})$, applying Lemma 2.1 to the last expression we get

$$\text{Var}(\bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5}) - \eta_{0.5}(u)) = O(\Gamma^{-2}) \quad (2.7)$$

for any $u \in R_{j,v}$. Now suppose $\mathcal{P}_{j,v}$ is the collection of paths corresponding to $\xi_{j,v}$. Denote, for any path P in $\mathcal{P}_{j,v}$, the quantity $\int_P \mathbb{E}(e_{\xi_{j,v}, P} e^{\gamma X(z)} | dz|$ as $q_{P, j, v}$. Using Fubini and (2.7) in a similar way as we used Fubini and Lemma 2.1 for (2.7), one gets

$$\text{Var}\left(\int_P \mathbb{E}(e_{\xi_{j,v}, P} e^{\gamma X(z)}) \eta(z) | dz| - q_{P, j, v} \bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5})\right) = O(q_{P, j, v}^2 \Gamma^{-2}), \quad (2.8)$$

for all $P \in \mathcal{P}_{j,v}$. Since

$$\bar{Z}(X, \xi_{j,v}, \eta_{0.5}; \gamma) = \sum_{P \in \mathcal{P}_{j,v}} \int_P \mathbb{E}(e_{\xi_{j,v}, P} e^{\gamma X(z)}) \eta(z) | dz|,$$

and $\sum_{P \in \mathcal{P}_{j,v}} q_{P,j,v} = L(\xi_{j,v}, X; \gamma) = 1$, (2.8) gives us

$$\text{Var}(\bar{Z}(X, \xi_{j,v}, \eta_{0.5}; \gamma) - \bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5})) = O(\Gamma^{-2}).$$

Denoting $\sum_{j \in [N]} \bar{Z}(\Gamma \partial_{\text{up}} R_{j,v}, \eta_{0.5})$ as $\bar{Z}_{v,N}$, we then have

$$\text{Var}\left(\sum_{j \in [N]} \bar{Z}(X, \xi_{j,w+iv}, \eta_{0.5}; \gamma) - \bar{Z}_{w+iv,N}\right) - \left(\sum_{j \in [N]} \bar{Z}(X, \xi_{j,w}, \eta_{0.5}; \gamma) - \bar{Z}_{w,N}\right) = O(N^2 \Gamma^{-2}).$$

In order to estimate $\text{Var}(\bar{Z}_{w+iv,N} - \bar{Z}_{w,N})$, on the other hand, we can use the definition of $\eta_{0.5}(v)$ in (2.2) and Fubini to obtain:

$$\begin{aligned} \text{Var}(\bar{Z}_{w+iv,N} - \bar{Z}_{w,N}) &= \frac{\Gamma^2}{\pi} \int_{[0, N/\Gamma]^2 \times [0.25, 1]} s^{-1} e^{-\frac{(x-z)^2}{2s}} (1 - e^{-\frac{v^2}{2s}}) dx dz ds \\ &= \Omega(\Gamma^2) \int_{[0.25, 1]} \int_{[0, N/\Gamma]} \int_{[0, N/\Gamma]} s^{-1} e^{-\frac{(x-z)^2}{2s}} dx dz ds \\ &= \Omega(N\Gamma), \end{aligned}$$

where in the second step we used $v \geq 0.1$ and in the final step the fact $\int_{[0,b]} e^{-ax^2} dx = \Omega_{a,b}(1)$. The last two displays yield the required bound on the standard deviation of $\sum_{j \in [N]} \bar{Z}(X, \xi_{j,w+iv}, \eta_{0.5}; \gamma) - \sum_{j \in [N]} \bar{Z}(X, \xi_{j,w}, \eta_{0.5}; \gamma)$. The bound on $\text{Var}(\sum_{j \in [N]} \bar{Z}(X, \xi_{j,w}, \eta_{0.5}; \gamma))$ follows from similar computations. \square

3 Inductive constructions for light paths

In this section we will discuss algorithms to construct light paths between the shorter boundaries of V^Γ when the underlying field is $\eta_{2^{-n}}$. Below we introduce some terms that will be used repeatedly.

A polypath ξ is said to *connect* two polypaths ξ' and ξ'' if ξ always intersects ξ' and ξ'' considered as subsets of \mathbb{R}^2 . More generally we say that the polypaths $\xi_1, \xi_2, \dots, \xi_k$ *form* or *define* a polypath if their union is always a connected subset of \mathbb{R}^2 . A *crossing* for a rectangle \mathcal{R} is any polypath ξ that stays entirely within \mathcal{R} and connects two shorter boundaries of \mathcal{R} .

Depending on the value of current scale n , we will use one of two different strategies for constructing a crossing cross_n for V^Γ . To be more precise, let $2a_n m_\Gamma \leq n < 2(a_n + 1)m_\Gamma$ where $2^{m_\Gamma} = \Gamma$ and $a_n \in \mathbb{N} \cup \{0\}$. We will use a simple strategy called *Strategy I* when $2a_n m_\Gamma \leq n < 2a_n m_\Gamma + 2m_\Gamma - 1$ and a different strategy called *Strategy II* otherwise. We detail these two strategies in separate subsections.

3.1 Strategy I

We will adopt an inductive approach. Consider the rectangle $0.5\iota + [0, \Gamma] \times [0, 2^{2a_n m_\Gamma - 1 - n}]$. Notice that this is same as $V_{n-2a_n m_\Gamma + 1}^{2^{n-2a_n m_\Gamma + 1} \Gamma; 0.5\iota}$. Subdivide it into non-overlapping translates of $V_{n-2a_n m_\Gamma + 1}^\Gamma$ and denote them by $R_{1;n-2a_n m_\Gamma + 1}, R_{2;n-2a_n m_\Gamma + 1}, \dots$ from left to right (see Figure 1).

Now suppose that for all $\ell \leq 2a_n m_\Gamma - 1$, we already have an algorithm $\mathcal{A}_{2a_n m_\Gamma - 1}$ that constructs a crossing through V^Γ and takes only the fields $\{\eta_{2^{-k}}\}_{k \in [\ell]}$ as input. Due to the scaling and translation invariance property of η we can then use $\mathcal{A}_{2a_n m_\Gamma - 1}$ to construct a crossing $\text{cross}_{j;n}$ through $R_{j;n-2a_n m_\Gamma + 1}$ using only the fields $\{\eta_{2^{-k}}^{2^{2a_n m_\Gamma - 1 - n}}\}_{n-2a_n m_\Gamma + 1 < k \leq n}$ as its input. Henceforth whenever we talk about applying \mathcal{A}_ℓ to construct a crossing at scale n , we will suppress the statement that the fields used to construct it are $\{\eta_{2^{-\ell}}^{2^{-(n-\ell')}}\}_{n-\ell' < \ell \leq n}$. The remaining job is to link the pair of crossings $\text{cross}_{j;n}$ and $\text{cross}_{j+1;n}$. This can

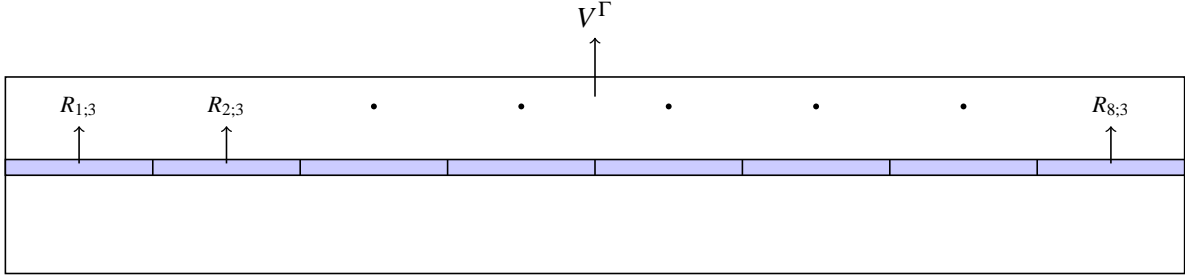


Figure 1 – The rectangles $R_{1;n-2a_n m_\Gamma+1}, R_{2;n-2a_n m_\Gamma+1}, \dots$. Here $n = 2a_n m_\Gamma + 2$.

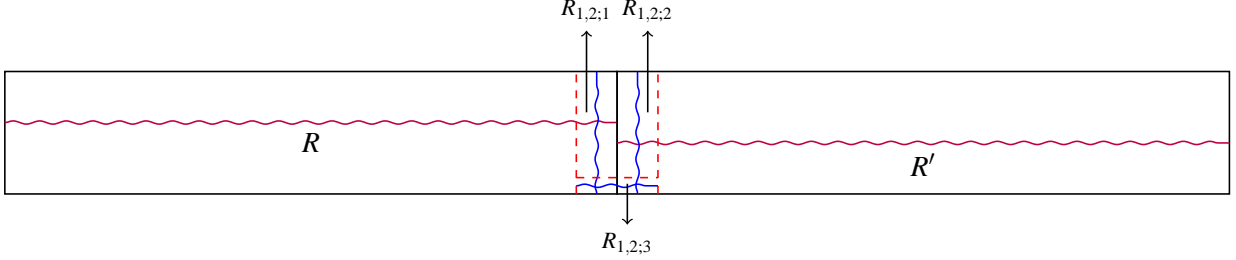


Figure 2 – **Tying cross_R and $\text{cross}_{R'}$.** The crossings cross_R and $\text{cross}_{R'}$ are indicated by purple lines. The two vertical blue lines indicate the crossings $\text{cross}^{*,R_{1,2,1}}$ (left) and $\text{cross}^{*,R_{1,2,2}}$ (right). The horizontal blue line indicates the crossing $\text{cross}^{*,R_{1,2,3}}$.

be done in a simple way which we call *tying* for convenience. We describe this technique in a general setting as it will be used several times in the future. The reader is referred to Figure 2 for an illustration.

Let $k \in [n-1]$. Consider two adjacent copies of V_k^Γ . Without any loss of generality (because of the rotational invariance property of $\eta_\delta^{\delta'}$), assume that their longer dimensions are aligned with the horizontal axis. Call the left one as $R = I \times J$ and the right one as $R' = I' \times J'$. We want to link two crossings cross_R and $\text{cross}_{R'}$ through R and R' respectively to build a crossing for $R \cup R'$. To this end define three additional rectangles $R_{1,2,1} = [r_I - 2^{-k-m_\Gamma}, r_I] \times J$, $R_{1,2,2} = [r_I, r_I + 2^{-k-m_\Gamma}] \times J$ and $R_{1,2,3} = [r_I - 2^{-k-m_\Gamma}, r_I + 2^{-k-m_\Gamma}] \times [\ell_J, \ell_J + 2^{-k-2m_\Gamma+1}]$, where ℓ_J and r_I are the left and right endpoints of J and I respectively. We use $\mathcal{A}_{n-k-m_\Gamma}$ to construct up-down crossings $\text{cross}_{R_{1,2,1}}$ and $\text{cross}_{R_{1,2,2}}$ for $R_{1,2,1}$ and $R_{1,2,2}$ respectively. Similarly we apply $\mathcal{A}_{n-k-2m_\Gamma+1}$ to construct a left-right crossing $\text{cross}_{R_{1,2,3}}$ through $R_{1,2,3}$. Let us also make it clear that \mathcal{A}_ℓ constructs a straight line connecting midpoints of the shorter boundary segments of V^Γ when $\ell \leq 0$. Finally notice that the union of crossings $\text{cross}_R, \text{cross}_{R'}, \text{cross}_{R_{1,2,1}}, \text{cross}_{R_{1,2,2}}$ and $\text{cross}_{R_{1,2,3}}$ is a crossing for the rectangle $R \cup R'$. We refer to this as the crossing obtained from tying cross_R and $\text{cross}_{R'}$.

Thus we tie together the sequence of crossings $\text{cross}_{1;n}, \text{cross}_{2;n}, \dots, \text{cross}_{2n-2a_n m_\Gamma+1;n}$ (i.e. every pair of successive crossings) to form cross_n . Figure 3 provides an illustration of this construction.

3.2 Strategy II

This is our main strategy which employs switching using sloped gadgets in order to build efficient crossings. Recall that $n = 2(a_n + 1)m_\Gamma - 1$ in this case. Unlike in Strategy I here we start with two strips $V_{2m_\Gamma}^{\Gamma^2;0.25\iota}$ and $V_{2m_\Gamma}^{\Gamma^2;0.75\iota}$. We subdivide $V_{2m_\Gamma}^{\Gamma^2;0.75\iota}$ and $V_{2m_\Gamma}^{\Gamma^2;0.25\iota}$ into non-overlapping translates of $V_{2m_\Gamma}^{\Gamma\beta}$ where β is the smallest power of 2 that is $\geq \gamma^{-2/3}$. Let us denote them as $R_{1,1}, R_{1,2}, \dots, R_{1,\Gamma/\beta}$ and $R_{2,1}, R_{2,2}, \dots, R_{2,\Gamma/\beta}$ respectively from left to right. Similarly one can subdivide each $R_{i,j}$ into non-

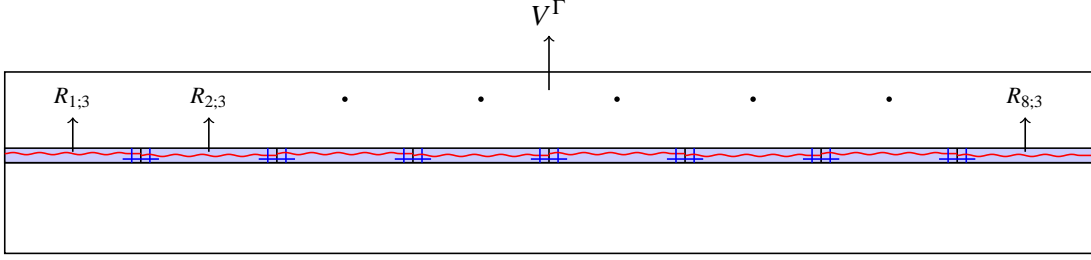


Figure 3 – **Construction of cross_n using Strategy I.** Here $n = 2a_n m_\Gamma + 2$. The red lines indicate the crossings $\text{cross}_{j,n}$'s while the blue lines indicate the crossings used for tying the pairs $(\text{cross}_{j,n}, \text{cross}_{j+1,n})$'s.

overlapping translates of $V_{2m_\Gamma}^{\Gamma^2}$ which we call as its *blocks*. See Figure 4 below for an illustration of this set-up. We can use $\mathcal{A}_{2a_n m_\Gamma - 1}$ to construct a crossing cross_R through each block R . Let $R_{i,j,\text{left}}$ and $R_{i,j,\text{right}}$

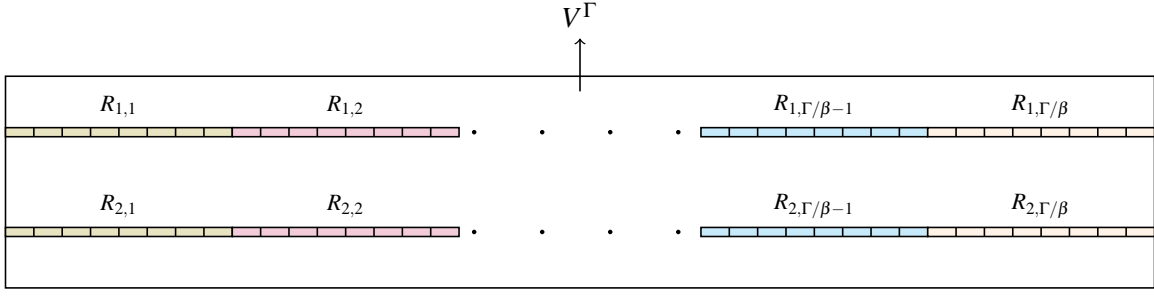


Figure 4 – **The rectangles $R_{i,j}$'s and its blocks.** In this (hypothetical) example each $R_{i,j}$ consists of 8 blocks and thus $\Gamma = 8$.

respectively denote the leftmost and rightmost blocks of $R_{i,j}$. We will construct a new sequence of crossings which, when tied, gives a polypath connecting $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Observe that, due to the choice of Γ and the fact $d_{\ell_2}(V_{2m_\Gamma}^{\Gamma^2; 0.25t}, V_{2m_\Gamma}^{\Gamma^2; 0.75t}) = \Omega(1)$, there exists an integer L_γ and a copy $S_{1,j}$ of $V_{2m_\Gamma}^{L_\gamma \Gamma}$ such that:

(I) The length of $S_{1,j}$ is at most $d_{\ell_2}(c_{R_{1,j,\text{left}}}, c_{R_{2,j,\text{right}}}) + 2/\Gamma$ where c_R denotes the center of a rectangle R .

(II) $S_{1,j}$, $R_{1,j,\text{left}}$ and $R_{2,j,\text{right}}$ are arranged as in Figure 5.

It is clear from the arrangement depicted in (II) (or in Figure 5 for that matter) that any crossing through $S_{1,j}$ intersects both $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Now subdivide $S_{1,j}$ into L_γ non-overlapping copies of $V_{2m_\Gamma}^\Gamma$ and construct a crossing through each one of them using $\mathcal{A}_{2a_n m_\Gamma - 1}$. Tying these crossings would then give a crossing of $S_{1,j}$ which connects $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Similarly we can construct a copy $S_{2,j}$ of $V_{2m_\Gamma}^{L_\gamma \Gamma}$ and a corresponding sequence of crossings which connect $\text{cross}_{R_{2,j,\text{left}}}$ and $\text{cross}_{R_{1,j,\text{right}}}$ after they are tied. The L_γ non-overlapping copies of $V_{2m_\Gamma}^\Gamma$ comprising $S_{i,j}$ are also called its *blocks*. Henceforth we will refer to the collection of blocks of $S_{i,j}$'s and $R_{i,j}$'s as Block_γ . We now have all the ingredients for defining our strategy which is essentially encoded by the numbers $i_j \in [2]$. Given these numbers, we define a collection \mathcal{C}_{a_n} of crossings as follows. If $i_j = i_{j+1}$, we include cross_R in \mathcal{C}_{a_n} for all the blocks R of $R_{i_j,j}$. Otherwise we include cross_R for all the blocks R of $S_{i_j,j}$ as well as $\text{cross}_{R_{i_j,j,\text{left}}}$ and $\text{cross}_{R_{3-i_j,j,\text{right}}}$ (notice that $3 - i_j$ switches 1 and 2). We refer to the collection of blocks included

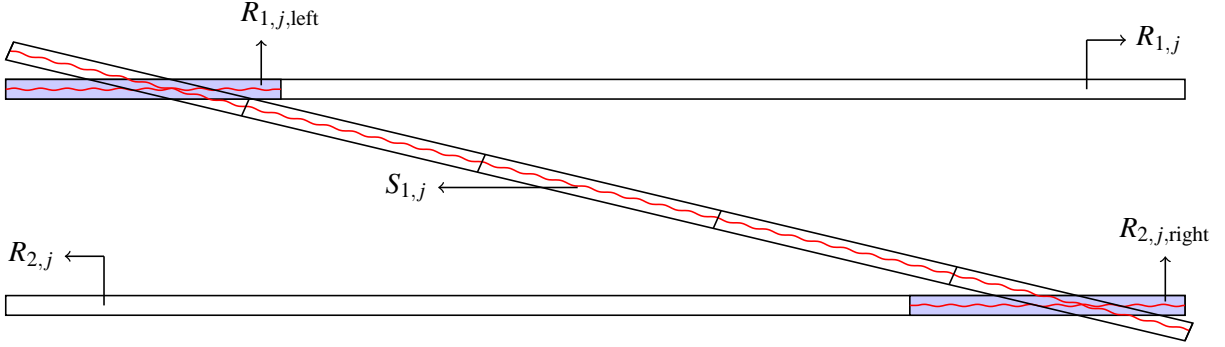


Figure 5 – **The rectangles $R_{1,j,\text{left}}$, $R_{2,j,\text{right}}$ and $S_{1,j}$.** Each of the five rectangles comprising $S_{1,j}$ is a copy of $V_{2m_\Gamma}^\Gamma$. Hence $L_\gamma = 5$ in this example. The red lines inside each rectangle indicate the corresponding crossings.

in \mathcal{C}_{a_n} from a “location” j as the *bridge* at that location. Unless there is a switch at location 1 (as $S_{i,1}$ can potentially intersect $\mathbb{R}^2 \setminus V^\Gamma$), the crossings in \mathcal{C}_{a_n} define a crossing for V^Γ after we tie every pair of crossings ($\text{cross}_R, \text{cross}_{R'}$) in \mathcal{C}_{a_n} for adjacent R, R' . See Figure 6 below for an illustration. The particular choice of i_j ’s will be determined by the field $\eta_{0.5}$ which we discuss in the next section.

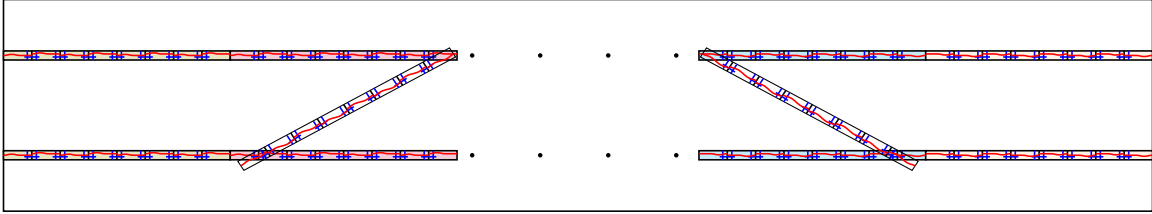


Figure 6 – **Construction of cross_n using Strategy II.** In this example $i_1 = i_2 = 2$ and $i_3 = 1$; $i_{\Gamma/\beta-1} = 1$ and $i_{\Gamma/\beta} = 2$. The red lines indicate the crossings in \mathcal{C}_{a_n} and the blue lines indicate the crossings used to tie them.

4 Multi-scale analysis on expected weight of crossings

Let $D_{\gamma,n}$ denote the total weight of cross_n computed with $\eta_{2^{-n}}$ as the underlying field and $d_{\gamma,n}$ denote its expectation. In Sections 4.1 through 4.3 we will derive recurrence relations involving $d_{\gamma,n}$ ’s for $n \in \mathbb{N}$. It is useful to recall at this point that $d_{\gamma,n} = \Gamma$ whenever $n \leq 0$. In Section 4.4 we show how these relations lead to a bound on $d_{\gamma,n}$.

4.1 Strategy I: a recurrence relation involving $d_{\gamma,n}$

We will assume $n > 0$. For convenience we use $[n]_\gamma$ to denote $n - 2a_n m_\Gamma + 1$. Let $D_{\gamma,n,\text{main}}$ denote the total weight of $\text{cross}_{1;n}, \text{cross}_{2;n}, \dots, \text{cross}_{[n]_\gamma;n}$ (see Section 3.1) and $D_{\gamma,n,\text{gadget}}$ denote the total weight of crossings used to tie them. These weights are all computed with respect to $\eta_{2^{-n}}$ and thus $D_{\gamma,n} = D_{\gamma,n,\text{main}} + D_{\gamma,n,\text{gadget}}$. Notice that the weight of $\text{cross}_{j;n}$ is $Z(\eta_{2^{-n}}^{2^{-[n]_\gamma}}, \text{cross}_{j;n}, e^{\gamma \eta_{2^{-[n]_\gamma}}}; \gamma)$ (see (2.4) for the definition of $Z(\cdot, \cdot, \cdot; \gamma)$). Hence from Fubini and the translation invariance property of η_δ we get

$$\mathbb{E} D_{\gamma,n,\text{main}} \leq \mathbb{E} e^{\gamma \eta_{2^{-[n]_\gamma}}(0)} 2^{[n]_\gamma} \frac{d_{\gamma, 2a_n m_\Gamma - 1}}{2^{[n]_\gamma}}, \quad (4.1)$$

where the divisor $2^{[n]\gamma}$ comes from scaling property (compare to the situation when $\gamma = 0$). From this point onwards any expression of the form “ $\frac{d_{\gamma,k}}{2^{n-k}}$ ” would implicitly mean that the divisor 2^{n-k} originates from a similar consideration. Now since $\text{Var}(\eta_\delta(0)) = O(\log \delta^{-1})$ and $m_\Gamma = O(\log \gamma^{-1})$, the last display gives us

$$\mathbb{E}D_{\gamma,n,\text{main}} = (1 + O(\gamma^2 \log \gamma^{-1}))d_{\gamma,2a_n m_\Gamma - 1}. \quad (4.2)$$

As to the estimation of $\mathbb{E}D_{\gamma,n,\text{gadget}}$, recall from Section 3.1 that we spend three crossings for tying the pair $(\text{cross}_{j;n}, \text{cross}_{j+1;n})$. Two of these are constructed using $\mathcal{A}_{(2a_n-1)m_\Gamma-1}$ and the other one using $\mathcal{A}_{2(a_n-1)m_\Gamma}$. Hence by a similar reasoning as used for (4.1), the expected weight of these crossings is given by

$$2\mathbb{E}e^{\gamma\eta_{2^{[n]\gamma}-m_\Gamma}(0)} \frac{d_{\gamma,(2a_n-1)m_\Gamma-1}}{2^{[n]\gamma+m_\Gamma}} + \mathbb{E}e^{\gamma\eta_{2^{[n]\gamma}-2m_\Gamma}(0)} \frac{d_{\gamma,2(a_n-1)m_\Gamma}}{2^{[n]\gamma+2m_\Gamma-1}}.$$

Since there are $2^{[n]\gamma} - 1$ many tyings, this implies (along with the variance bounds given by (2.3))

$$\mathbb{E}D_{\gamma,n,\text{gadget}} \leq (1 + O(\gamma^2 \log \gamma^{-1}))(2\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma-1} + \Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma}).$$

Combined with (4.2), the last inequality gives us

$$d_{\gamma,n} \leq (1 + O(\gamma^2 \log \gamma^{-1}))(d_{\gamma,2a_n m_\Gamma - 1} + 2\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma-1} + \Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma}). \quad (4.3)$$

4.2 Strategy II: choosing the particular strategy

As in Section 4.1, we begin with a decomposition of $D_{\gamma,n}$ into two components. To this end denote by $D_{\gamma,n,\text{main}}$ the total weight of crossings in \mathcal{C}_{a_n} where $n = 2a_n m_\Gamma + 2m_\Gamma - 1$. The other component $D_{\gamma,n,\text{gadget}}$ is the total weight of gadgets that we use to tie pairs of crossings $(\text{cross}_R, \text{cross}_{R'})$ in \mathcal{C}_{a_n} for adjacent R, R' (see Section 3.2). All the weights are computed with respect to the field $\eta_{2^{-n}}$. $D_{\gamma,n,\text{main}}$ is the major component and will inform our choice of strategy.

We, in fact, devise our strategy based on an approximate expression of $\mathbb{E}(D_{\gamma,n,\text{main}} | \eta_{0.5})$. For this we need to analyze $D_{\gamma,n,\text{main};j}$ which is the combined weight of crossings through all the blocks in the bridge at location j . In our analysis we rely heavily on the fact that our strategy is determined by $\eta_{0.5}$. Also along the way we make several approximations that will be justified in a later subsection. Let us begin with the case $i_j = i_{j+1}$. In this case

$$\mathbb{E}(D_{\gamma,n,\text{main};j} | \eta_{0.5}) = \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{0.5}, \text{cross}_R, e^{\gamma\eta_{0.5}}; \gamma).$$

Now we replace $\eta_{2^{-n}}^{0.5}$ in the above expression with $\eta_{2^{-n}}^{\Gamma^{-2}}$ which results in

$$\text{approx}_{j,1} = \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma^{-2}}, \text{cross}_R, e^{\gamma\eta_{0.5}}; \gamma).$$

We further approximate $e^{\gamma\eta_{0.5}(z)}$ with $1 + \gamma\eta_{0.5}(z)$ and obtain a new expression as follows (recall the definitions (2.5) and (2.6)):

$$\begin{aligned} \text{approx}_{j,2} &= \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma^{-2}}, \text{cross}_R, 1 + \gamma\eta_{0.5}; \gamma) \\ &= \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} L(\text{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}; \gamma) + \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma^{-2}}, \text{cross}_R, \gamma\eta_{0.5}; \gamma) \\ &= \beta\Gamma \frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2} + \gamma \sum_{R \in \text{Block}_\gamma, R \subseteq R_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma^{-2}}, \text{cross}_R, \eta_{0.5}; \gamma) \\ &= \beta\Gamma^{-1}d_{\gamma,2a_n m_\Gamma - 1} + \bar{Z}_{\gamma,n,i_j,j} \end{aligned}$$

where $\bar{Z}_{\gamma,n,i_j,j} = \gamma \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma-2}, \text{cross}_R, \eta_{0.5}; \gamma)$. Thus there is a “deterministic” part and a “random” part in $\text{approx}_{j,2}$. The small magnitude of γ is crucial for these approximations. When $i_j \neq i_{j+1}$, i.e. there is a switch at the location j , deriving $\text{approx}_{j,2}$ requires slightly more work. In this case

$$\mathbb{E}(D_{\gamma,n,\text{main};j} | \eta_{0.5}) = \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j} \cup R_{i_j,j,\text{left}} \cup R_{3-i_j,j,\text{right}}} \bar{Z}(\eta_{2^{-n}}^{0.5}, \text{cross}_R, e^{\gamma \eta_{0.5}}; \gamma).$$

Similarly as before we ignore the contribution from $\eta_{\Gamma-2}^{0.5}$ and the higher order terms in $e^{\gamma \eta_{0.5}(z)}$ to obtain

$$\text{approx}_{j,1} = \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j} \cup R_{i_j,j,\text{left}} \cup R_{3-i_j,j,\text{right}}} \bar{Z}(\eta_{2^{-n}}^{\Gamma-2}, \text{cross}_R, e^{\gamma \eta_{0.5}}; \gamma),$$

and

$$\text{approx}_{j,2} = \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j} \cup R_{i_j,j,\text{left}} \cup R_{3-i_j,j,\text{right}}} L(\text{cross}_R, \eta_{2^{-n}}^{\Gamma-2}; \gamma) + \bar{Z}'_{\gamma,n,i_j,j},$$

where

$$\bar{Z}'_{\gamma,n,i_j,j} = \gamma \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j} \cup R_{i_j,j,\text{left}} \cup R_{3-i_j,j,\text{right}}} \bar{Z}(\eta_{2^{-n}}^{\Gamma-2}, \text{cross}_R, \eta_{0.5}; \gamma).$$

Recall from Section 3.2 that the total number of blocks in the bridge in this case is $L_\gamma + 2$. From property (I) of $S_{i_j,j}$ and the elementary fact $\sqrt{1+x^2} = 1 + \frac{x^2}{2} + o(x^2)$ as $x \rightarrow 0$, we get

$$\frac{L_\gamma}{\Gamma} = \beta + \frac{C'_{\gamma,n}}{\beta},$$

for some $C'_{\gamma,n} = \Theta(1)$. Hence the deterministic part in $\text{approx}_{j,2}$ is

$$\beta \Gamma \frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2} + C''_{\gamma,n} \frac{\Gamma}{\beta} \frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2},$$

where again $C''_{\gamma,n} = \Theta(1)$. Writing the random part $\bar{Z}'_{\gamma,n,i_j,j}$ as

$$\bar{Z}'_{\gamma,n,i_j,j} = \bar{Z}_{\gamma,n,i_j,j} + \text{Loss}_{\gamma,n,i_j,j}, \quad (4.4)$$

we obtain in this case

$$\text{approx}_{j,2} = \beta \Gamma \frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2} + C''_{\gamma,n} \frac{\Gamma}{\beta} \frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2} + \bar{Z}_{\gamma,n,i_j,j} + \text{Loss}_{\gamma,n,i_j,j}.$$

Now from Lemma 2.2 we have

$$\begin{aligned} \text{Var}\left(\gamma \sum_{R \in \text{Block}_\gamma, R \subseteq S_{i_j,j}} \bar{Z}(\eta_{2^{-n}}^{\Gamma-2}, \text{cross}_R, \eta_{0.5}; \gamma)\right) &= \gamma^2 \Gamma^2 \frac{d_{\gamma,2a_n m_\Gamma - 1}^2}{\Gamma^4} O(\beta + \beta^2 \Gamma^{-2}) \\ &= O(\Gamma^2 \gamma^2 \beta) \frac{d_{\gamma,2a_n m_\Gamma - 1}^2}{\Gamma^4}. \end{aligned}$$

The same bound holds for $\text{Var}(\bar{Z}_{\gamma,n,i_j,j})$ and (obviously) for $\text{Var}(\gamma \bar{Z}(\eta_{2^{-n}}^{\Gamma-2}, \text{cross}_R, \eta_{0.5}; \gamma))$ when $R = R_{i_j,j,\text{left}}$ or $R_{i_j,j,\text{right}}$. Thus from (4.4) we get

$$\text{Var}(\text{Loss}_{\gamma,n,i_j,j}) = O(\Gamma^2 \gamma^2 \beta) \frac{d_{\gamma,2a_n m_\Gamma - 1}^2}{\Gamma^4}.$$

As $\text{Loss}_{\gamma,n,i,j}$'s are centered Gaussian variables, the previous bound implies

$$\sum_{i \in [2]} \mathbb{E}(\text{Loss}_{\gamma,n,i,j}^+) = O(\Gamma \gamma \sqrt{\beta}) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma^2}.$$

Incorporating this bound into the expression for $\text{approx}_{j,2}$ we get the following upper bound on expectation of $\text{approx}_2 = \sum_{j \in [\Gamma/\beta]} \text{approx}_{j,2}$.

$$\mathbb{E}(\text{approx}_2) \leq d_{\gamma,2a_n m_{\Gamma}-1} + \mathbb{E} \sum_{j \in [\Gamma/\beta]} \bar{Z}_{\gamma,n,i_j,j} + C_{\gamma,n}(\beta^{-1} + \gamma \sqrt{\beta}) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma} N_{\text{switch}}, \quad (4.5)$$

where $C_{\gamma,n} = \Theta(1)$ and N_{switch} is total number of “potential” switching locations (deterministic). Since $\bar{Z}_{\gamma,n,i_j,j}$'s are centered, we can write

$$\mathbb{E} \sum_{j \in [\Gamma/\beta]} \bar{Z}_{\gamma,n,i_j,j} = \frac{1}{2} \mathbb{E} \sum_{j \in [\Gamma/\beta]} (-1)^{i_j+1} (\bar{Z}_{\gamma,n,1,j} - \bar{Z}_{\gamma,n,2,j}).$$

Hence we choose our strategy so that it gives a small value of the following expectation:

$$E_{\gamma,n} = \mathbb{E} \left(\frac{1}{2} \sum_{j \in [\Gamma/\beta]} (-1)^{i_j+1} \Delta \bar{Z}_{\gamma,n,j} + C_{\gamma,n}(\beta^{-1} + \gamma \sqrt{\beta}) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma} N_{\text{switch}} \right), \quad (4.6)$$

where $\Delta \bar{Z}_{\gamma,n,j} = \bar{Z}_{\gamma,n,1,j} - \bar{Z}_{\gamma,n,2,j}$. From Lemma 2.2 we can deduce that for any $1 \leq j_1 \leq j_2 \leq \lceil \Gamma/\beta \rceil$,

$$\begin{aligned} \text{Var} \left(\sum_{j_1 \leq j \leq j_2} \frac{1}{2} \Delta \bar{Z}_{\gamma,n,j} \right) &= \Omega(\gamma^2) \frac{d_{\gamma,2a_n m_{\Gamma}-1}^2}{\Gamma^4} (j_2 - j_1 + 1) \beta \Gamma^2 \left(1 - O\left(\frac{\sqrt{(j_2 - j_1 + 1)\beta}}{\Gamma}\right) \right) \\ &\geq c_2 \gamma^2 (j_2 - j_1 + 1) \beta \frac{d_{\gamma,2a_n m_{\Gamma}-1}^2}{\Gamma^2}. \end{aligned}$$

As a consequence we have

$$\mathbb{E} \left| \sum_{j_1 \leq j \leq j_2} \frac{1}{2} \Delta \bar{Z}_{\gamma,n,j} \right| \geq 2C_{\gamma,n}(\beta^{-1} + \gamma \sqrt{\beta}) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma} \quad (4.7)$$

whenever

$$j_2 - j_1 + 1 \geq \frac{4C_{\gamma,n}^2(\beta^{-1} + \gamma \sqrt{\beta})^2}{\frac{2}{\pi} \gamma^2 \beta} = N'_{\gamma,n}. \quad (4.8)$$

Here we use the simple fact that $\mathbb{E}|Z| = \sqrt{\frac{2}{\pi}}$ for a standard Gaussian Z . Let $N_{\gamma,n}$ be the smallest power of 2 that is $\geq N'_{\gamma,n}$. We are now ready to define our strategy. Set

$$i_j = \begin{cases} 0 & \text{if } \sum_{(k_j-1)N_{\gamma,n}+1 \leq j' \leq k_j N_{\gamma,n}} \Delta \bar{Z}_{\gamma,n,j'} > 0, \\ 1 & \text{otherwise,} \end{cases}$$

where $k_j \in \mathbb{N}$ is such that $(k_j - 1)N_{\gamma,n} + 1 \leq j \leq k_j N_{\gamma,n}$. It then follows from (4.6), (4.7) and (4.8), and the choice of β as $\Theta(\gamma^{-2/3})$ that

$$\begin{aligned} E_{\gamma,n} &\leq \frac{\Gamma}{N_{\gamma,n} \beta} \times -C_{\gamma,n}(\beta^{-1} + \gamma \sqrt{\beta}) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma} = -\Omega\left(\frac{\gamma^2}{\beta^{-1} + \gamma \sqrt{\beta}}\right) d_{\gamma,2a_n m_{\Gamma}-1} \\ &= -\Omega(\gamma^{4/3}) d_{\gamma,2a_n m_{\Gamma}-1}. \end{aligned} \quad (4.9)$$

Notice that this strategy ensures $i_1 = i_2$ i.e. there is no switch at location 1 which implies we get a “legitimate” crossing (see the discussions at the end of Section 3.2).

4.3 Strategy II: a recurrence relation involving $d_{\gamma,n}$

Let us first estimate the expected errors that we made in every stage of approximation in the previous subsection. Denote the sum $\sum_{j \in [\Gamma/\beta]} \text{approx}_{j,1}$ as approx_1 . Since the choice of crossings in \mathcal{C}_{a_n} is independent with $\eta_{\Gamma-2}^{0.5}$, from Fubini and translation invariance of η we get

$$\mathbb{E}D_{\gamma,n,\text{main}} = \mathbb{E}e^{\gamma\eta_{\Gamma-2}^{0.5}(0)} \mathbb{E}(\text{approx}_1) = (1 + O(\gamma^2 \log \gamma^{-1})) \mathbb{E}(\text{approx}_1).$$

Next we take care of the approximation of approx_1 with approx_2 . Since $e^x \geq 1 + x$, it follows that $\text{approx}_1 \geq \text{approx}_2$. On the other hand, a reasoning similar to the one used for last display gives us

$$\mathbb{E}(\text{approx}_1 - \text{approx}_2) \leq \mathbb{E}(e^{\gamma\eta_{0.5}(0)} - 1 - \gamma\eta_{0.5}(0)) \frac{d_{\gamma,2a_n m_{\Gamma}-1}}{\Gamma^2} |\text{Block}_{\gamma}|.$$

It is straightforward that $|\text{Block}_{\gamma}| = O(\Gamma^2)$ and hence

$$\mathbb{E}(\text{approx}_1 - \text{approx}_2) \leq O(\gamma^2) d_{\gamma,2a_n m_{\Gamma}-1}.$$

Since $\mathbb{E}(\text{approx}_2) \leq d_{\gamma,2a_n m_{\Gamma}-1} + E_{\gamma,n}$ (see (4.5), (4.6)), the bounds from the previous displays and (4.9) together imply

$$\mathbb{E}D_{\gamma,n,\text{main}} \leq d_{\gamma,2a_n m_{\Gamma}-1} (1 - \Omega(\gamma^{4/3})). \quad (4.10)$$

It only remains to deal with $\mathbb{E}D_{\gamma,n,\text{gadget}}$. In fact the argument that we used to bound $\mathbb{E}D_{\gamma,n,\text{gadget}}$ for Strategy I can be applied directly in this case to obtain

$$\mathbb{E}D_{\gamma,n,\text{gadget}} \leq (2 \frac{d_{\gamma,(2a_n-1)m_{\Gamma}-1}}{\Gamma^3} \mathbb{E}e^{\gamma\eta_{\Gamma-3}(0)} + \frac{d_{\gamma,2(a_n-1)m_{\Gamma}}}{\Gamma^4} \mathbb{E}e^{\gamma\eta_{2\Gamma-4}(0)}) |\text{Block}_{\gamma}|.$$

which implies

$$\mathbb{E}D_{\gamma,n,\text{gadget}} = O(1)(\Gamma^{-1} d_{\gamma,(2a_n-1)m_{\Gamma}-1} + \Gamma^{-2} d_{\gamma,2(a_n-1)m_{\Gamma}}). \quad (4.11)$$

Finally (4.10) and (4.11) give us

$$d_{\gamma,n} \leq d_{\gamma,2a_n m_{\Gamma}-1} (1 - \Omega(\gamma^{4/3})) + O(1)(\Gamma^{-1} d_{\gamma,(2a_n-1)m_{\Gamma}-1} + \Gamma^{-2} d_{\gamma,2(a_n-1)m_{\Gamma}}). \quad (4.12)$$

4.4 Upper bound on $d_{\gamma,n}$

We will use the recursion relations (4.3), (4.12) and an induction argument to derive an upper bound on $d_{\gamma,n}$. To this end let C_2 be a positive, absolute constant (from (4.12)) such that

$$d_{\gamma,2(a+1)m_{\Gamma}-1} \leq d_{\gamma,2am_{\Gamma}-1} (1 - C_2 \gamma^{4/3}) + O(1)(\Gamma^{-1} d_{\gamma,(2a-1)m_{\Gamma}-1} + \Gamma^{-2} d_{\gamma,2(a-1)m_{\Gamma}}), \quad (4.13)$$

for all $a \geq 0$. Fixing an $a \in \mathbb{N} \cup \{0\}$, we formulate our induction hypotheses as:

- (a) $d_{\gamma,2am_{\Gamma}-1} \leq \Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^a$.
- (b) $d_{\gamma,n} \leq 2\Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^{a_n+1}$ for all $n < 2am_{\Gamma}$.

Hypotheses (a) and (b) obviously hold for $a = 0$ since $d_{\gamma,n} = \Gamma$ for $n \leq 0$. Now combined with (4.13) and the fact $\Gamma \geq \gamma^{-2}$, these two hypotheses imply

$$\begin{aligned} d_{\gamma,2(a+1)m_{\Gamma}} &\leq \Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^a (1 - C_2 \gamma^{4/3}) + O(\gamma^2) \Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^{a-1} \\ &= \Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^{a+1} (1 - \Omega(\gamma^{4/3}) + O(\gamma^2)) \leq \Gamma (1 - \frac{C_2 \gamma^{4/3}}{2})^{a+1}. \end{aligned}$$

On the other hand for $2am_\Gamma \leq n < 2(a+1)m_\Gamma$, we can apply (4.3) and hypotheses (a), (b) to obtain

$$\begin{aligned} d_{\gamma, 2(a+1)m_\Gamma} &\leq (1 + O(\gamma^2 \log \gamma^{-1})) \Gamma \left(\left(1 - \frac{C_2 \gamma^{4/3}}{2}\right)^a + O(\gamma^2) \left(1 - \frac{C_2 \gamma^{4/3}}{2}\right)^{a-1} \right) \\ &= \Gamma \left(1 - \frac{C_2 \gamma^{4/3}}{2}\right)^{a+1} (1 + O(\gamma^{4/3})) \leq 2\Gamma \left(1 - \frac{C_2 \gamma^{4/3}}{2}\right)^{a+1}. \end{aligned}$$

Thus by induction it follows that

$$d_{\gamma, n} \leq 2\Gamma \left(1 - \frac{C_2 \gamma^{4/3}}{2}\right)^{a_n+1}, \quad (4.14)$$

for all $n \geq 0$.

5 Proof of Theorem 1.1

For the purpose of proving Theorem 1.1, we can identify h_δ with its *white noise decomposition* given in (2.1). Since this representation involves some special functions, it would be helpful to have convenient notations for them. To this end we denote $p(s; v, w) - p_D(s; v, w)$ as $\bar{p}_D(s; v, w)$. Also for any function f defined on $\mathbb{R}^+ \times D_\varepsilon \times D_\varepsilon$ and $\delta \leq \varepsilon$, the function $f^\delta(\cdot; v, \cdot)$ denotes the average $\int_{\partial B_\delta(v)} f(\cdot; v', \cdot) \mu_\delta^v(dv')$. Now notice that we can decompose the difference $h_\delta(v) - \eta_\delta(v)$ into four components as follows:

$$h_\delta(v) - \eta_\delta(v) = G_{v;1} + G_{v;2} + G_{v;3} + G_{v;4},$$

where

$$\begin{aligned} G_{v;1} &= \int_{D \times [1, \infty)} p_D^\delta(s/2; v, w) W(dw, ds), \quad G_{v;2} = \int_{D \times (0, \delta^2]} p_D^\delta(s/2; v, w) W(dw, ds), \\ G_{v;3} &= - \int_{\mathbb{R}^2 \times [\delta^2, 1]} \bar{p}_D^\delta(s/2; v, w) W(dw, ds) \text{ and } G_{v;4} = \int_{\mathbb{R}^2 \times [\delta^2, 1]} (p^\delta(s/2; v, w) - p(s/2; v, w)) W(dw, ds). \end{aligned}$$

We will show that the variance of each component is $O_{D, \varepsilon}(1)$. Let us begin with $\text{Var}(G_{v;1})$. Observe that

$$\begin{aligned} \text{Var}(G_{v;1}) &= \int_{[1, \infty)} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p_D(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds \\ &= \int_{[1, \infty)} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p(s; v', v'') \mathcal{P}^D(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds, \end{aligned} \quad (5.1)$$

where $\mathcal{P}^D(s; v', v'')$ is the probability that a (two dimensional) Brownian bridge of duration s remains in D . Since squared absolute norm of a standard Brownian motion at time t is distributed as an exponential variable with mean $2t$, a simple computation gives us

$$\mathcal{P}^D(s; v', v'') = O(1) \left(1 - e^{-O\left(\frac{(d_{\ell_2}(v', \partial D) + |v' - v''|)^2}{s}\right)}\right).$$

Plugging this into (5.1) we get

$$\begin{aligned} \text{Var}(G_{v;1}) &= O(1) \int_{[1, \infty)} s^{-1} \left(1 - e^{-O\left(\frac{(d_{\ell_2}(v', \partial D) + |v' - v''|)^2}{s}\right)}\right) ds \\ &= O(1) \int_{[1, \infty)} s^{-2} (d_{\ell_2}(v', \partial D) + |v' - v''|)^2 ds = O(\text{diam}(D)^2). \end{aligned}$$

Next is $\text{Var}(G_{v;2})$ which can be evaluated as

$$\begin{aligned}
\text{Var}(G_{v;2}) &= \int_{(0,\delta^2]} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p_D(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds \\
&\leq \int_{(0,\delta^2]} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds \\
&= (2\pi)^{-2} \int_{(0,\delta^2]} s^{-1} \int_{[0,2\pi]} e^{-\frac{\delta^2(1-\cos\theta)}{s}} d\theta ds = O(1) \int_{(0,\delta^2]} e^{-\frac{\Omega(\delta^2)}{s}} s^{-1} ds = O(1).
\end{aligned}$$

For $\text{Var}(G_{v;3})$ we start with an upper bound:

$$\begin{aligned}
\text{Var}(G_{v;3}) &\leq \int_{[\delta^2,1]} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} \bar{p}_D(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds \\
&= O(1) \int_{[\delta^2,1]} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p(s; v', v'') \bar{\mathcal{P}}^D(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds, \quad (5.2)
\end{aligned}$$

where $\bar{\mathcal{P}}^D(s; v', v'')$ is the probability that a Brownian bridge of duration s hits ∂D . Like $\mathcal{P}^D(s; v', v'')$, we can use tail probabilities of appropriate exponentials to bound this as

$$\bar{\mathcal{P}}^D(s; v', v'') = O(1) e^{-\Omega\left(\frac{(d_{\ell_2}(v', \partial D) - |v' - v''|)^2}{s}\right)}.$$

(5.2) and the previous bound together imply

$$\text{Var}(G_{v;3}) \leq O(1) \int_{[\delta^2,1]} e^{-\Omega\left(\frac{(d_{\ell_2}(v', \partial D) - |v' - v''|)^2}{s}\right)} ds = O_\varepsilon(1),$$

We are only left with $\text{Var}(G_{v;4})$ now. Notice that

$$\begin{aligned}
\text{Var}(G_{v;4}) &= \int_{[\delta^2,1]} \int_{\partial B_\delta(v) \times \partial B_\delta(v)} p(s; v', v'') \mu_\delta^v(dv') \mu_\delta^v(dv'') ds - 2 \int_{[\delta^2,1]} \int_{\partial B_\delta(v)} p(s; v', v) \mu_\delta^v(dv') ds \\
&\quad + \int_{[\delta^2,1]} p(s; v, v) ds \\
&= I_1 - 2I_2 + I_3.
\end{aligned}$$

Since $p(s; v', v'') \leq (2\pi)^{-1} s^{-1}$, it follows that I_1 and I_3 are bounded above by $(2\pi)^{-1} \int_{[\delta^2,1]} s^{-1} ds$. On the other hand,

$$I_2 = (2\pi)^{-1} \int_{[\delta^2,1]} e^{-\frac{\delta^2}{s}} s^{-1} ds \geq (2\pi)^{-1} \int_{[\delta^2,1]} (1 - \delta^2 s^{-1}) s^{-1} ds.$$

Putting all these estimates together we get $\text{Var}(G_{v;4}) = O(1)$. Thus $\text{Var}(h_\delta(v) - \eta_\delta(v)) = O_{D,\varepsilon}(1)$ for all $v \in D_\varepsilon$. In addition we claim that

$$\text{Var}((h_\delta(v) - \eta_\delta(v)) - (h_\delta(w) - \eta_\delta(w))) = O\left(\frac{|v - w|}{\delta}\right) \quad (5.3)$$

for all $v, w \in D_\varepsilon$ such that $|v - w| \leq \delta$. Thus, by Dudley's entropy bound on the supremum of a Gaussian process (see, e.g., [1, Theorem 4.1]) and Gaussian concentration inequality (see e.g., [29, Equation (7.4), Theorem 7.1]) we deduce that

$$\mathbb{P}\left(\max_{v \in V} (h_\delta(v) - \eta_\delta(v)) > C_3 \sqrt{\log \delta^{-1}} + x\right) = e^{-\Omega_{D,\varepsilon}(x^2)}, \quad (5.4)$$

for all $x \geq 0$. We will verify (5.3) shortly, but before that let us show how (5.4) leads to a proof of Theorem 1.1. To this end define, for $v, w \in V$, $D_{\eta, \gamma, \delta}(v, w) = \inf_P \int_P e^{\gamma \eta_\delta(z)} |dz|$ where P ranges over all piecewise smooth paths in V connecting v and w . Also denote by $D_{h, \gamma, \delta}^{\text{straight}}(v, w)$ the weight of the straight line joining v and w when the underlying field is h_δ . The following is straightforward:

$$D_{\gamma, \delta}(v, w) \leq e^{O(\gamma \sqrt{\log \delta^{-1}})} D_{V, \eta, \gamma, \delta}(v, w) \mathbf{1}_{E_V} + D_{h, \gamma, \delta}^{\text{straight}}(v, w) \mathbf{1}_{E_V^c}, \quad (5.5)$$

where $M_V = \max_{v \in V} (h_\delta(v) - \eta_\delta(v))$ and E_V is the event $\{M_V \leq (C_3 + 1) \sqrt{\log \delta^{-1}}\}$. Now consider the unique integer n such that $2^{-n-1} < \delta \leq 2^{-n}$. Let $d_{\gamma, n}^*$ the expected weight of cross_n computed with respect to η_δ . From (4.14), independent increment of η and the $O(1)$ bound on $\text{Var}(\eta_{\delta'}(v))$ for $\delta' \leq 1$, it follows that

$$d_{\gamma, n}^* = O(\Gamma) \delta^{\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})}. \quad (5.6)$$

Now notice that any point $v \in V$ can be connected to any boundary segment of V by constructing crossings through a sequence of rectangles $R_{1,v}, R_{2,v}, \dots, R_{K,v}$, with geometrically growing size such that (a) each $R_{i,v}$ is a subset of V , (b) $K = O(\log \delta^{-1})$ and (c) the ratio of longer to shorter dimension of each $R_{i,v}$ is at most Γ . If we connect v and w to each of the four boundary segments in this way, it is easy to see that the union of all such crossings must contain a path between v and w . Using the translation invariance and scaling property of η along with (5.7) we can bound the expected total weight of these crossings as follows

$$\mathbb{E} D_{\eta, \gamma, \delta}(v, w) = O(\Gamma) \delta^{\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})}.$$

As to $D_{h, \gamma, \delta}^{\text{straight}}(v, w) \mathbf{1}_{E_V^c}$, we can use (5.4) and Cauchy-Schwarz inequality to obtain

$$\mathbb{E} D_{\eta, \gamma, \delta}(v, w) \leq e^{-\Omega_{D, \varepsilon}(\log \delta^{-1})} \sqrt{\mathbb{E} (D_{h, \gamma, \delta}^{\text{straight}}(v, w))^2}. \quad (5.7)$$

Since $\text{Var}(h_\delta(v) - \eta_\delta(v)) = O_{D, \varepsilon}(1)$ and $\text{Var}(\eta_\delta(v)) = O(\log \delta^{-1})$, we get from Fubini

$$\mathbb{E} (D_{h, \gamma, \delta}^{\text{straight}}(v, w))^2 = \int_{[0,1]^2} e^{\gamma(h_\delta(x+0.5t) + h_\delta(y+0.5t))} dx dy \leq O_{D, \varepsilon}(1) \delta^{-O(\gamma^2)}. \quad (5.8)$$

The last four displays together imply

$$\mathbb{E} D_{\gamma, \delta}(v, w) = O_{\gamma, D, \varepsilon}(1) \delta^{\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})},$$

which proves Theorem 1.1.

It only remains to verify (5.3). Since

$$(h_\delta(v) - \eta_\delta(v)) - (h_\delta(w) - \eta_\delta(w)) = (h_\delta(v) - h_\delta(w)) - (\eta_\delta(v) - \eta_\delta(w)),$$

it suffices to prove similar bounds for each of $\text{Var}(h_\delta(v) - h_\delta(w))$ and $\text{Var}(\eta_\delta(v) - \eta_\delta(w))$. The latter can be obtained from Lemma 2.1. The bound on $\text{Var}(h_\delta(v) - h_\delta(w))$ was derived (in a more general set-up) in the proof of [26, Proposition 2.1].

6 Proof of Theorem 1.2

Let us first introduce some notations. \mathbb{D} is the open, unit disk centered at the origin. For any closed ball $B \equiv c_B + r\overline{\mathbb{D}}$, we let \tilde{B} denote the open ball $c_B + 4r\overline{\mathbb{D}}$. Let h^* be a GFF with Dirichlet boundary conditions on some bounded domain \mathcal{D} with smooth boundary. For any $\delta > 0$ and $v \in \mathcal{D}$ such that $d_{\ell_2}(v, \partial\mathcal{D}) < \delta$, we denote the average of h^* along the circle $v + \delta\partial\mathbb{D}$ as $h_\delta^*(v)$.

Now consider a GFF $h^\mathcal{D}$ on \mathcal{D} with Dirichlet boundary conditions. If $\tilde{B} \subseteq \mathcal{D}$, then *Markov property* (see [40, Section 2.6] or [7, Theorem 1.17]) of GFF states that $h^\mathcal{D} = h^{\mathcal{D}, \tilde{B}} + \varphi^{\mathcal{D}, B}$, where

- $h^{\mathcal{D}, \tilde{B}}$ is a GFF on \tilde{B} with Dirichlet boundary conditions (= 0 outside \tilde{B}).
- $\varphi^{\mathcal{D}, B}$ is harmonic on \tilde{B} .
- $h^{\mathcal{D}, \tilde{B}}, \varphi^{\mathcal{D}, B}$ are independent.

This decomposition has a useful consequence for us as follows. Since $\varphi^{\mathcal{D}, B}$ is harmonic on \tilde{B} , we get

$$h_\delta^\mathcal{D}(v) = h_\delta^{\mathcal{D}, \tilde{B}}(v) + \varphi^{\mathcal{D}, B}(v), \quad (6.1)$$

for all $v \in \overline{B}^{2*} = c_B + 2r\overline{\mathbb{D}}$ and $\delta \in (0, r]$. The process $\{h_\delta^{\mathcal{D}, \tilde{B}}(v) : v \in \overline{B}^{2*}, 0 < \delta \leq r\}$ is independent with $\{\varphi^{\mathcal{D}, B}(v) : v \in \overline{B}^{2*}\}$ and also with $h_{\delta'}^\mathcal{D}(w)$ for $w \in \mathcal{D} \setminus \tilde{B}, \delta' < d_{\ell_2}(w, \tilde{B})$. The following lemma shows that the field $\varphi^{\mathcal{D}, B}$ is smooth on \overline{B}^{2*} .

Lemma 6.1. *Let $B \equiv c_B + r\overline{\mathbb{D}} \subseteq \mathcal{D}$ be a closed ball such that $\tilde{B} \subseteq \mathcal{D}$. Then we have for all $v, w \in \overline{B}^{2*}$*

$$\text{Var}(\varphi^{\mathcal{D}, B}(v) - \varphi^{\mathcal{D}, B}(w)) = O\left(\frac{|v - w|}{r}\right).$$

Also,

$$\sup_{v \in \overline{B}^{2*}} \text{Var}(h_r^\mathcal{D}(c_B) - \varphi^{\mathcal{D}, B}(v)) = O(1).$$

Proof. Since $h_r^{\tilde{B}}$ and $\varphi^{\mathcal{D}, B}$ are independent, we get from (6.1)

$$\text{Var}(\varphi^{\mathcal{D}, B}(v) - \varphi^{\mathcal{D}, B}(w)) \leq \text{Var}(h_r^\mathcal{D}(v) - h_r^\mathcal{D}(w)),$$

for all $v, w \in \overline{B}^{2*}$. But we know (see the proof of [26, Proposition 2.1])

$$\text{Var}(h_r^\mathcal{D}(v) - h_r^\mathcal{D}(w)) = O\left(\frac{|v - w|}{r}\right),$$

which gives the required bound on $\text{Var}(\varphi^{\mathcal{D}, B}(v) - \varphi^{\mathcal{D}, B}(w))$. For the second part, notice that

$$\text{Var}(h_r^\mathcal{D}(c_B) - \varphi^{\mathcal{D}, B}(c_B)) = \text{Var}(h_r^{\mathcal{D}, \tilde{B}}(c_B)).$$

Thus it suffices to prove $\text{Var}(h_r^{\mathcal{D}, \tilde{B}}(c_B)) = O(1)$ in view of the bound on $\text{Var}(\varphi^{\mathcal{D}, B}(v) - \varphi^{\mathcal{D}, B}(w))$. But $h_r^{\mathcal{D}, \tilde{B}}(c_B)$ is identically distributed as $h_{0.25}^\mathbb{D}(0)$ by the scale and translation invariance of GFF and hence $\text{Var}(h_r^{\mathcal{D}, \tilde{B}}(c_B))$ is a finite constant (see the discussions in [40, Section 2.1] and also [7, Theorem 1.9]). \square

Now consider a Radon measure μ on \mathfrak{D} and some $\delta > 0$. We call a closed Euclidean ball $B \subseteq \mathfrak{D}$ with a rational center as a (μ, δ) -ball if $\mu(B) \leq \delta^2$. For any compact $A \subseteq \mathfrak{D}$, let $N(\mu, \delta, A)$ denote the minimum number of (μ, δ) -balls required to cover A . Our next proposition provides a crude upper bound on the *second moment* of $N(M_\gamma^\mathbb{D}, \delta, A)$ (see (1.2) for the definition of $M_\gamma^\mathbb{D}$) when A is a segment inside \mathbb{D} . We remark that the KPZ relation proved in [21] gives the sharp exponent on the first moment of $N(M_\gamma^\mathbb{D}, \delta, A)$.

Proposition 6.2. *Let \mathcal{H} denote the straight line segment joining -0.25 and 0.25 . For any $\delta \in (0, 1)$, we can find a collection of $(M_\gamma^\mathbb{D}, \delta)$ -balls $\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})$ such that*

- (a) *Balls in $\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})$ cover \mathcal{H} .*
- (b) *All the balls in $\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})$ are contained in $0.25\overline{\mathbb{D}}$.*
- (c) *For some C_4 ,*

$$\mathbb{E}(|\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})|^2) = O_\gamma(\delta^{-2-C_4\gamma}),$$

Proof. For each $k \in \mathbb{N}$, let \mathcal{B}_k denote the collection of all (closed) balls of radius 2^{-k-1} whose centers lie in the set $\{-\frac{1}{4} + 2^{-k-1}, -\frac{1}{4} + 3 \cdot 2^{-k-1}, \dots, \frac{1}{4} - 2^{-k-1}\}$. The balls in \mathcal{B}_k are nested in a natural way. In particular any ball B in \mathcal{B}_k has a unique *parent* $B^{k'}$ in $\mathcal{B}_{k'}$ (where $k' \leq k$) such that $B \subseteq B^{k'}$. Let $\mathcal{B}(M_\gamma^\mathbb{D}, k, \delta)$ denote the collection of balls in \mathcal{B}_k with $M_\gamma^\mathbb{D}$ volume $> \delta^2$. We include a $(M_\gamma^\mathbb{D}, \delta)$ -ball $B \in \mathcal{B}_k$ in $\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})$ if the $M_\gamma^\mathbb{D}$ volume of the *most recent* parent of B is bigger than δ^2 . Since the measure $M_\gamma^\mathbb{D}$ is a.s. is finite and has no atoms (see [21] and [7, Theorem 2.1]), it follows that $\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})$ satisfies condition (a) (and obviously (b)). It also follows from the construction that

$$|\mathcal{S}(M_\gamma^\mathbb{D}, \delta, \mathcal{H})| \leq 2\delta^{-1-C_4'\gamma} + \sum_{k > (1+C_4'\gamma)\log_2 \delta^{-1}} |\mathcal{B}(M_\gamma^\mathbb{D}, k, \delta)|, \quad (6.2)$$

where $C_4' > 1$ is some fixed constant to be specified later. Observing that $|\mathcal{B}(M_\gamma^\mathbb{D}, k, \delta)|$ is the total number of balls in \mathcal{B}_k with $M_\gamma^\mathbb{D}$ volume $> \delta^2$ a naive bound can be obtained as

$$\begin{aligned} \left(\sum_{k > (1+C_4'\gamma)\log_2 \delta^{-1}} |\mathcal{B}(M_\gamma^\mathbb{D}, k, \delta)| \right)^2 &\leq \sum_{k > (1+C_4'\gamma)\log_2 \delta^{-1}} \sum_{B \in \mathcal{B}_k} 2 \sum_{k' \leq k} \sum_{B' \in \mathcal{B}_{k'}} \mathbf{1}_{\{M_\gamma^\mathbb{D}(B) > \delta^2, M_\gamma^\mathbb{D}(B') > \delta^2\}} \\ &\leq \sum_{k > (1+C_4'\gamma)\log_2 \delta^{-1}} \sum_{B \in \mathcal{B}_k} 2 \sum_{k' \leq k} \sum_{B' \in \mathcal{B}_{k'}} \mathbf{1}_{\{M_\gamma^\mathbb{D}(B) > \delta^2\}} \\ &\leq \sum_{k > (1+C_4'\gamma)\log_2 \delta^{-1}} 2^{k+1} \sum_{B \in \mathcal{B}_k} \mathbf{1}_{\{M_\gamma^\mathbb{D}(B) > \delta^2\}}. \end{aligned} \quad (6.3)$$

Next we compute the probability that any given ball $B \equiv c_B + 2^{-k}\mathbb{D}$ in \mathcal{B}_k has $M_\gamma^\mathbb{D}$ volume at least δ^2 . Since $M_\gamma^\mathbb{D}$ (or $M_{B,k}^\mathbb{D}$) is the weak limit of measures $M_{\gamma,n}^\mathbb{D}$'s (respectively $M_{B,k}^\mathbb{D}$'s) defined in (1.2), we have

$$M_\gamma^\mathbb{D}(B) \leq 4^{-k} 2^{-\frac{k\pi^{-1}\gamma^2}{2}} e^{\gamma h_{2^{-k}}^\mathbb{D}(c_B)} \times e^{\gamma M_{B,k}^\mathbb{D}} \times 4^k M_{\gamma,B}^\mathbb{D}(B), \quad (6.4)$$

where $M_{B,k} = \max_{v \in \overline{B}^{2^*}} (\varphi^{\mathbb{D},B}(v) - h_{2^{-k}}^\mathbb{D}(c_B))$ and $M_{\gamma,B}^\mathbb{D}$ is the LQG measure on \tilde{B} obtained from $h^{\mathbb{D},B}$. From the scale and translation invariance property of GFF (see, e.g., [40, Page 8] and [7, Theorem 1.19]) it follows that $4^k M_{\gamma,B}^\mathbb{D}(B)$ is identically distributed as $M_\gamma^\mathbb{D}(\frac{1}{4}\mathbb{D})$. Using this observation and (6.4) we can write,

$$\mathbb{P}(M_\gamma^\mathbb{D}(B) > \delta^2) \leq \mathbb{P}(h_{2^{-k}}^\mathbb{D}(c_B) \geq \frac{2}{3\gamma} \log(\delta 2^k)) + \mathbb{P}(M_{B,k}^\mathbb{D} \geq \frac{2}{3\gamma} \log(\delta 2^k)) + \mathbb{P}(M_\gamma^\mathbb{D}(\frac{1}{4}\mathbb{D}) \geq 4^{k/3} \delta^{2/3}).$$

Since $\text{Var}(h_{2^{-k}}^{\mathbb{D}}(c_B)) = k \log 2 + C_B$ for $|C_B| = \Theta(1)$ and $\delta^{1+C_4\gamma} > 2^{-k}$, the first term on the right hand side of the previous display can be bounded as

$$\mathbb{P}(\eta_{2^{-k}}(c_B) \geq \frac{2}{3\gamma} \log(\delta 2^k)) \leq \mathbb{P}\left(Z \geq \frac{C'_4 k \log 2}{3\sqrt{k \log 2 + C_{B,\gamma}}}\right) \leq e^{-C_4'^2 \Omega(k) \log 2} = 2^{-C_4'^2 \Omega(k)}.$$

Here Z is a standard Gaussian variable. Thus we can choose C'_4 big enough so that the bound above becomes $< 2^{-10k}$. From Lemma 6.1 we know that

$$\max_{v \in B} \text{Var}(\varphi^{\mathbb{D},B}(v) - h_{2^{-k}}^{\mathbb{D}}(c_B)) = O(1) \text{ and } \text{Var}(\varphi^{\mathbb{D},B}(v) - \varphi^{\mathbb{D},B}(w)) \leq O\left(\frac{|v-w|}{2^{-k}}\right)$$

for all $v, w \in B$. Hence, similar to the derivation of (5.4), by Dudley's entropy bound and Gaussian concentration inequality we get for all sufficiently large k

$$\mathbb{P}(M_{B,k}^{\mathbb{D}} \geq \frac{2}{3\gamma} \log(\delta 2^k)) \leq 2^{-10k}.$$

The only remaining term is $\mathbb{P}(M_{\gamma}^{\mathbb{D}}(\frac{1}{4}\overline{\mathbb{D}}) \geq 4^{k/3} \delta^{2/3})$. In order to bound this probability we will use the fact that $\mathbb{E}(M_{\gamma}^{\mathbb{D}}(\frac{1}{4}\overline{\mathbb{D}}))^4 < \infty$ (see [27] and also [38, Theorem 2.11 and Theorem 5.5]). Hence by Chebychev's inequality

$$\mathbb{P}(M_{\gamma}^{\mathbb{D}}(\frac{1}{4}\overline{\mathbb{D}}) \geq 4^{k/3} \delta^{2/3}) = O_{\gamma}(\delta^{-8/3}) 2^{-8k/3}.$$

Plugging the last three estimates into the expression for the upper bound on $\mathbb{P}(M_{\gamma}(B) \geq \delta^2)$ we get

$$\mathbb{P}(M_{\gamma}^{\mathbb{D}}(B) > \delta^2) \leq O_{\gamma}(\delta^{-8/3}) 2^{-8k/3}.$$

Taking expectation on both sides in (6.3) and using the bound above one gets:

$$\begin{aligned} \mathbb{E}\left(\sum_{k > (1+C_4'\gamma) \log_2 \delta^{-1}} |\mathcal{B}(M_{\gamma}, k, \delta)|\right)^2 &\leq \sum_{k > (1+C_4'\gamma) \log_2 \delta^{-1}} 2^{2k+1} O_{\gamma}(\delta^{-8/3}) 2^{-8k/3} \\ &= O_{\gamma}(\delta^{-8/3}) \sum_{k > (1+C_4'\gamma) \log_2 \delta^{-1}} 2^{-2k/3} = O_{\gamma}(\delta^{-8/3}) \delta^{2/3+8\gamma} = O_{\gamma}(\delta^{-2+8\gamma}). \end{aligned}$$

The lemma follows from this bound and (6.2) for $C_4 = \max(C'_4, 8)$. \square

The proof of Proposition 6.2 can be easily adapted to accommodate the following set-ups.

Corollary 6.3. *Let $S \subseteq V$ be a closed square of length 2^{-k} whose vertices lie in $2^{-k}\mathbb{Z}^2$. Then for any $\delta \in (0, 2^{-k})$ we have*

$$\mathbb{E}N(M_{\gamma}^{\mathbb{D}}, \delta, S)^2 = O_{\gamma,D,\varepsilon}((2^k \delta)^{-4-O(\gamma)} 2^{kO(\gamma)}).$$

Now, given a $\delta \in (0, 1)$ and $v, w \in V$, we will construct a collection of (closed) balls $\mathcal{S}(\delta, v, w)$ such that every ball in $\mathcal{S}(\delta, v, w)$ has $M_{\gamma}^{\mathbb{D}}$ measure at most δ and the union of balls in $\mathcal{S}(\delta, v, w)$ contains a path connecting v and w . Thus it would suffice to show

$$\mathbb{E}|\mathcal{S}(\delta, v, w)| = O_{\gamma,D,\varepsilon}(1) \delta^{-1+\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})}$$

for proving Theorem 1.2. Before we describe the construction of $\mathcal{S}(\delta, v, w)$, we need to discuss a related construction which will be very useful. To this end define, for any fixed $k > 4$, the set \mathcal{D}_k as $2^{-(k-3)}\mathbb{Z}^2 \cap V$. We can treat \mathcal{D}_k as a subgraph of the lattice $2^{-(k-3)}\mathbb{Z}^2$. The centers of the squares in \mathcal{D}_k (i.e. the squares of side length $2^{-(k-3)}$ with vertices in \mathcal{D}_k) form another set $\mathcal{D}_k^* \subseteq V$ which will be treated as the dual graph of \mathcal{D}_k . We can define a LFPP metric $D_{\gamma,k}^*(\cdot, \cdot)$ on \mathcal{D}_k^* in a similar way as in (1.3) with $h_{2^{-k}}$ as the underlying field and $\gamma(1+C_4\gamma)/2$ as the inverse temperature parameter (see Proposition 6.2 for the definition of C_4). The next lemma is a consequence of our proof of Theorem 1.1

Lemma 6.4. For all $\gamma > 0$ sufficiently small,

$$\max_{u, u' \in \mathcal{D}_k^*} \mathbb{E} D_{\gamma, k}^*(u, u') = O_{\gamma, D, \varepsilon}(1) 2^{k(1 - \Omega(\gamma^{4/3}/\log \gamma^{-1}))}.$$

Proof. Let V_k^* denote the square $[2^{-(k-3)}, 1 - 2^{-(k-3)}]^2$ so that $\mathcal{D}_k^* \subseteq V_k^*$. Following the proof of Theorem 1.1 in the last section, we get a *fixed*, finite collection $\mathcal{P}_k(u, u')$ of piecewise smooth paths in V_k^* between $u, u' \in V_k^*$ and a (randomly chosen) simple, piecewise smooth path $P_{k, \gamma}(u, u') \in \mathcal{P}_k$ such that

$$\mathbb{E} \left(\int_{P_{k, \gamma}(u, u')} e^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)/2} |dz| \right) = O_{\gamma, D, \varepsilon}(2^{-k\Omega(\gamma^{4/3}/\log \gamma^{-1})}). \quad (6.5)$$

In order to create a lattice path (i.e. in \mathcal{D}_k^*) between u and u' from $P_{k, \gamma}(u, u')$ we follow a simple procedure. Starting from the initial point $p_{k, \gamma, 0}$ of $P_{k, \gamma}(u, u')$, wait until it exits the smallest square S_0 satisfying (a) $p_{k, \gamma, 0} \in S_0$, (b) $d_{\ell_2}(p_{n, 0}, \partial S_0) \geq 2^{-(k-3)}$ and (c) the vertices of S_0 are in \mathcal{D}_k^* . Repeat the same procedure with the exit point of $P_{k, \gamma}(u, u')$ and continue until it reaches u' . At the end of this procedure we will get a sequence of squares S_0, S_1, \dots , where each S_i has diameter at most $32^{-(k-3)}$ and the vertices of S_i 's contain a lattice path $P_{k, \gamma}^*(u, u')$ between u and u' . Now let us recall from the previous subsection that

$$\max_{z, z' \in V_k^*, |z - z'| \leq 2^{-(k-3)}} \text{Var}(h_{2^{-k}}(z) - h_{2^{-k}}(z')) = O(1), \text{ and } \max_{z \in V_k^*} \text{Var}(h_{2^{-k}}(z)) = O(k) + O_{D, \varepsilon}(1).$$

Then from the arguments involving the extreme values of Gaussian processes as used for (5.4), we can find C_5 such that

$$\mathbb{P} \left(\max_{z, z' \in V_k^*, |z - z'| \leq 2^{-(k-3)}} (h_{2^{-k}}(z) - h_{2^{-k}}(z')) \geq C_5 \sqrt{k} + x \right) = e^{-\Omega_{D, \varepsilon}(x^2)}, \quad (6.6)$$

for all $x \geq 0$. Now define an event E_k as

$$E_k = \left\{ \max_{z, z' \in V_k^*, |z - z'| \leq 2^{-(k-3)}} (h_{2^{-k}}(z) - h_{2^{-k}}(z')) \leq (C_5 + 1) \sqrt{k} \right\}.$$

As the euclidean length of $P_{k, \gamma}(u, u')$ inside each S_i is $\Omega(2^{-k})$, from (6.5) it follows that

$$\begin{aligned} \mathbb{E} \left(\left(\sum_{z \in P_{k, \gamma}^*(u, u')} e^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)} \right) \mathbf{1}_{E_k} \right) &= O(2^k) e^{(C_5+1)\sqrt{k}} O_{\gamma, D, \varepsilon}(2^{-k\Omega(\gamma^{4/3}/\log \gamma^{-1})}) \\ &= O_{\gamma, D, \varepsilon}(2^{k(1 - \Omega(\gamma^{4/3}/\log \gamma^{-1}))}). \end{aligned}$$

On the other hand, from (6.6) and Cauchy-Schwarz inequality (similar to (5.7) and (5.8)) we obtain

$$\mathbb{E} \left(\left(\sum_{z \in P_k(u, u')} e^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)} \right) \mathbf{1}_{E_k^c} \right) = O_{D, \varepsilon}(2^k) 2^{-k(\Omega_{D, \varepsilon}(1) - O(\gamma^2))} = O_{D, \varepsilon}(2^{k(1 - \Omega_{D, \varepsilon}(1))}),$$

where $P_k(u, u')$ is the shortest path between u and u' in the graph \mathcal{D}_k^* . Choosing $P_{k, \gamma}^*(u, u')$ and $P_k(u, u')$ on E_k and E_k^c respectively as a lattice path between u and u' , we get the desired bound on $\mathbb{E} D_{\gamma, k}^*(u, u')$ from the previous two displays. \square

We will call the path minimizing $D_{\gamma, k}^*(u, u')$ as the (γ, k) -geodesic between u and u' . Now given v, w in V , we pick squares v_k and w_k in \mathcal{D}_k that contain v and w respectively. There are several ways to do this and we follow an arbitrary but fixed convention. Define $\mathcal{S}^*(k, v, w)$ as the collection of squares in \mathcal{D}_k which correspond to the points in the (γ, k) -geodesic between $c([v]_k)$ and $c([w]_k)$ in \mathbb{D}_k^* . Here $c([v]_k)$ and $c([w]_k)$ are the centers of squares $[v]_k$ and $[w]_k$ respectively. Thus $\mathcal{S}^*(k, v, w)$ is actually a *chain of squares* connecting v and w (see Figure 7). An important observation is the following.

Observation 6.5. *The euclidean distance between the boundary of any square in $\mathcal{S}^*(k, v, w)$ and \mathcal{D}_k^* is at least $2^{-(k-2)}$.*

Given $S \in \mathcal{S}^*(k, v, w)$ that is not $[v]_k$ or $[w]_k$, divide each boundary segment of S into 16 segments (with disjoint interiors) of length $2^{-(k+1)}$. For any such segment T , let B_T denote the closed ball of radius $2^{-(k+2)}$ centered at the midpoint of T . Thus T is a diameter segment of B_T . Cover T with the minimum possible number of $(M_{\gamma, B_T}^D, \delta e^{-\gamma h_{2^{-k}}^D(c(S))/2} e^{-C_6 \gamma \sqrt{k \log 2}})$ -balls contained in B_T where M_{γ, B_T}^D is the LQG measure on \tilde{B}_T constructed from h^{D, \tilde{B}_T} , $c(S)$ is the center of S and C_6 is an absolute constant to be specified later. Denote the collection of all such balls from all the segments of ∂S as $\mathcal{S}(S, \delta)$. If $S = [v]_k$ or $[w]_k$, we simply cover S with minimum possible number of (M_γ^D, δ) -balls and include them in $\mathcal{S}(S, \delta)$. Finally define

$$\mathcal{S}^{**}(k, \delta, v, w) = \bigcup_{S \in \mathcal{S}^*(k, v, w)} \mathcal{S}(S, \delta).$$

It is clear that the union of balls in $\mathcal{S}^{**}(k, \delta, v, w)$ contains a path between v and w . Figure 7 gives an illustration of this construction.

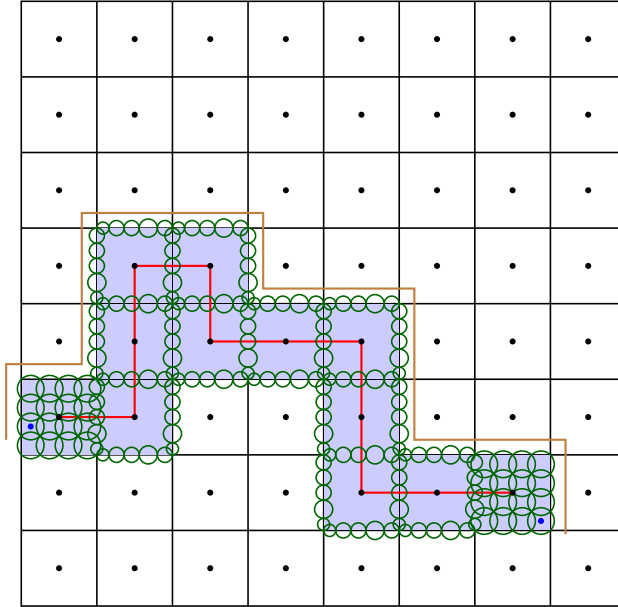


Figure 7 – **An instance of $\mathcal{S}^{**}(k, \delta, v, w)$.** Squares in $\mathcal{S}^*(k, \delta, v, w)$ are filled with light blue color. The black dotted points lie in \mathcal{D}_k^* . v (left) and w (right) are indicated as blue dotted points. The red (lattice) path is the LFPP path between $c([v]_k)$ and $c([w]_k)$. The green circles indicate the balls in $\mathcal{S}^{**}(k, \delta, v, w)$. Balls that lie parallel to the brown segments define a chain of ball connecting v and w .

We will now describe the construction of $\mathcal{S}(\delta, v, w)$. By Lemma 6.1, the bounds on $\text{Var}(h_{\delta^*}(u) - h_{\delta^*}(u'))$ and $\text{Var}(h_{\delta^*}(v))$, and tail estimates as used in (5.4) and (6.6), we get C_6 such that for all k sufficiently large (depending on D, ε)

- (a) $\mathbb{P}(\min_{u \in V} h_{2^{-k}}(u) < -2C_6 k \log 2) \leq 2^{-3k}$ and
- (b) $\mathbb{P}(\max_S \max_B \max_{v \in \tilde{B}^*} (\varphi^{D, B}(v) - h_{2^{-k}}^D(c(S))) > 2C_6 \sqrt{k \log 2}) \leq 2^{-3k},$

where in (b), S ranges over all squares in \mathcal{D}_k and B ranges over all balls of radius $2^{-(k+2)}$ around S that we described in the last paragraph. Choose δ' as the smallest number of the form 2^{-k} (where $k \in \mathbb{N}$) such that $\delta' \geq \delta^{1-2C_6\gamma}$. Now if

$$\min_{u \in V} h_{\delta'}(u) < -2C_6 \log \delta'^{-1}$$

or if

$$\max_S \max_B \max_{v \in \bar{B}^{2*}} (\varphi^{D,B}(v) - h_{2^{-k}}^D(c(S))) > 2C_6 \sqrt{k \log 2},$$

(we call the union of these two events as E_δ) simply cover the straight line segment joining v and w with the minimum possible number of (M_γ^D, δ) -balls. Otherwise (i.e. on E_δ^c) set $\mathcal{S}(\delta, v, w) = \mathcal{S}^{**}(k', \delta, v, w)$ where $\delta' = 2^{-k'}$. Notice that $\mathcal{S}^{**}(k, \delta, v, w)$ is a valid choice for $\mathcal{S}(\delta, v, w)$ on E_δ^c as

$$M_\gamma^D(A) \leq e^{\gamma \max_S \max_B \max_{v \in \bar{B}^{2*}} (\varphi^{D,B}(v) - h_{2^{-k}}^D(c(S)))} e^{\gamma h_{2^{-k}}^D(c(S))} M_{\gamma, B_T}^D(A)$$

for all B_T and all compact $A \subseteq B_T$ (this again follows from the definition of LQG measure as a weak limit).

Upper bound on $\mathbb{E}(|\mathcal{S}(\delta, v, w)|)$: Let us denote the σ -field generated by $\{h_{\delta'}(v) : v \in \mathcal{D}_{k'}^*\}$ as \mathfrak{F}_δ and the event $\{\min_{v \in \mathcal{D}_{k'}^*} h_{\delta'}(v) \geq -2C_6 \log \delta'^{-1}\}$ as F_δ . We then have

$$\begin{aligned} \mathbb{E}(|\mathcal{S}(\delta, v, w)| | \mathfrak{F}_\delta) &\leq \sum_{S \in \mathcal{S}^{**}(k', v, w)} \sum_T \mathbb{E}(N^*(M_{\gamma, B_T}^D, \delta e^{-\gamma h_{\delta'}^D(c(S))/2} e^{-C_6 \gamma \sqrt{k' \log 2}}, T) | \mathfrak{F}_\delta) \mathbf{1}_{F_\delta} \\ &+ \mathbb{E}(N(M_\gamma^D, \delta, \overline{vw}) \mathbf{1}_{E_{\delta'}} | \mathfrak{F}_\delta) + \mathbb{E}(N(M_\gamma^D, \delta, [v]_{k'}) | \mathfrak{F}_\delta) + \mathbb{E}(N(M_\gamma^D, \delta, [w]_{k'}) | \mathfrak{F}_\delta), \end{aligned}$$

where T ranges over all the 16×4 segments of ∂S and $N^*(M_{\gamma, B_T}^D, r, T)$ is the minimum possible number of (M_{γ, B_T}^D, r) -balls contained in B_T that are required to cover T . By the Markov property of GFF (see the discussions around (6.1)) and Observation 6.5 it follows that M_{γ, B_T}^D is identically distributed as $M_\gamma^{\tilde{B}_T}$ and is independent with \mathfrak{F}_δ . The latter is identically distributed as $\frac{\delta'^2}{16} M_\gamma^{\mathbb{D}}$ by scale and translation invariance property of GFF. Also on F_δ ,

$$\delta e^{-\gamma h_{\delta'}^D(c(S))/2} e^{-C_6 \gamma \sqrt{k' \log 2}} < \delta \delta'^{-C_6 \gamma} \leq \delta'^{(1-2C_6 \gamma)^{-1} - C_6 \gamma} < \delta'.$$

We can then apply Proposition 6.2 to the first term in the right hand side of the previous display to get

$$\begin{aligned} \mathbb{E}|\mathcal{S}(\delta, v, w)| &\leq O_\gamma((\delta/\delta')^{-1-C_4\gamma/2}) e^{C_6 \gamma \sqrt{\log 2 k'} (1+C_4\gamma/2)} \mathbb{E}\left(\sum_{S \in \mathcal{S}^{**}(k', v, w)} e^{\frac{\gamma(1+C_4\gamma)h_{\delta'}(c(S))}{2}}\right) \\ &+ \mathbb{E}N(M_\gamma^D, \delta, \overline{vw}) \mathbf{1}_{E_{\delta'}} + \mathbb{E}N(M_\gamma^D, \delta, [v]_{k'}) + \mathbb{E}N(M_\gamma^D, \delta, [w]_{k'}). \end{aligned}$$

The first term on the right hand side equals, by Lemma 6.4

$$\begin{aligned} &O_\gamma(\delta^{-2C_6\gamma(1+C_4\gamma/2)}) O_{\gamma, D, \varepsilon}(1) \delta'^{-1+\Omega(\gamma^{4/3}/\log \gamma^{-1})} = O_{\gamma, D, \varepsilon}(\delta^{-2C_6\gamma(1+C_4\gamma/2)}) \delta^{2C_6\gamma} \delta^{-1+\Omega(\gamma^{4/3}/\log \gamma^{-1})} \\ &= O_{\gamma, D, \varepsilon}(\delta^{-1+\Omega(\gamma^{4/3}/\log \gamma^{-1})}). \end{aligned}$$

The second term is $O(1)$ as a consequence of bounds (a), (b), Corollary 6.3 and Cauchy-Schwarz inequality (similar to (5.7) and (5.8)). The last two terms are $O_{\gamma, D, \varepsilon}(\delta^{-O(\gamma)})$ by Corollary 6.3. Adding up these four terms, we get the required bound on $\mathbb{E}|\mathcal{S}(\delta, v, w)|$.

7 Adapting to discrete GFF

Let $N = 2^n$, $V_N^\Gamma \equiv ([0, \Gamma N - 1] \times [0, N - 1]) \cap \mathbb{Z}^2$ and $V_N^{\Gamma, \varepsilon} = ([-\lfloor \varepsilon \Gamma N \rfloor, \Gamma N + \lfloor \varepsilon \Gamma N \rfloor - 1] \times [-\lfloor \varepsilon N \rfloor, N + \lfloor \varepsilon N \rfloor + 1]) \cap \mathbb{Z}^2$. Consider a discrete Gaussian free field $\{\eta_{\gamma, N}(v) : v \in V_N^{\Gamma, \varepsilon}\}$ on $V_N^{\Gamma, \varepsilon}$ with Dirichlet boundary condition. By interpolation we can extend $\eta_{\gamma, N}$ to a continuous field on the rectangle $[-\varepsilon \Gamma N, (1 + \varepsilon) \Gamma N] \times [-\varepsilon N, (1 + \varepsilon) N]$. After appropriate scaling we then get a continuous Gaussian field $\tilde{\eta}_{\gamma, N}$ on the domain $V^{\Gamma, \varepsilon} = (-\varepsilon \Gamma, (1 + \varepsilon) \Gamma) \times (-\varepsilon, (1 + \varepsilon))$. It is clear that we need to find a suitable decomposition for the covariance kernel of $\eta_{\gamma, N}$ in order to get a decomposition of $\tilde{\eta}_{\gamma, N}$ similar to the white noise decomposition of η_δ . The covariance between $\eta_{\gamma, N}(v)$ and $\eta_{\gamma, N}(w)$ is given by the simple random walk Green function $G_{V_N^{\Gamma, \varepsilon}}(v, w)$. There is a simple representation of $G_{V_N^{\Gamma, \varepsilon}}(\cdot, \cdot)$ as a sum of simple random walk probabilities. However here we represent it in terms of *lazy* simple random walk probabilities for reasons that would become clear shortly. To this end we write

$$G_{V_N^{\Gamma, \varepsilon}}(v, w) = \frac{1}{2} \sum_{k=0}^{\infty} \mathbb{P}^v(S_k = w, \tau_{\gamma, \varepsilon} > k),$$

where $\{S_k\}_{k \geq 0}$ is a lazy simple random walk on \mathbb{Z}^2 i.e. it stays put for each step with probability $\frac{1}{2}$ and jumps to each of its four neighbors with probability $\frac{1}{8}$, \mathbb{P}^v is the measure corresponding to the random walk starting from v and $\tau_{\gamma, \varepsilon}$ is the first time the random walk hits $\partial V_N^{\Gamma, \varepsilon}$. Emulating our approach to the approximation of circle average process with η_δ , we replace $\tau_{\gamma, \varepsilon}$ in the above representation with the order of its expectation i.e. N^2 (on V_N^Γ , of course) and obtain a new kernel:

$$K_N(v, w) = \frac{1}{2} \sum_{k=1}^{N^2-1} \mathbb{P}^v(S_k = w).$$

Notice that, thanks to the laziness of S_k , each matrix $(\mathbb{P}^v(S_t = w))_{v, w \in V_N^{\Gamma, \varepsilon}}$ is non-negative definite. The similarity between this expression and the integral representation of $\text{Cov}(\eta_\delta(v), \eta_\delta(w))$ prompts the following decomposition of $K_N(\cdot, \cdot)$:

$$K_N(v, w) = \sum_{k' \in [n]} \frac{1}{2} \sum_{4^{k'-1} \leq k < 4^{k'}} \mathbb{P}^v(S_k = w) = \sum_{k' \in [n]} K_{N, k'}(v, w).$$

Hence we can “approximate” $\tilde{\eta}_{\gamma, N}$ with a sum of independent, stationary fields $\Delta \tilde{\eta}_{N, k'}$ on V_N where the covariance kernel of $\Delta \tilde{\eta}_{N, k'}$ is “given” by $K_{N, k'}$. Denote $\tilde{\eta}_{N, k'} = \sum_{k'' \in [k']} \Delta \eta_{N, k''}$. It is immediate that the sequence of fields $\tilde{\eta}_{N, k'}$ ’s are stationary and have independent increments. Using standard results on discrete planar random walk and local central limit theorem estimates (see, e.g., Chapters 2 and 4 in [28]) one can also prove the following properties:

- (a) $\text{Var}(\Delta \tilde{\eta}_{N, k'}(v)) = O(1)$ and $\text{Var}(\Delta \tilde{\eta}_{N, k'}(v) - \Delta \tilde{\eta}_{N, k'}(w)) = 4^{n-k'} O(|v - w|^2)$ for all v, w . Compare this to Lemma 2.1.
- (b) For any straight line segment \mathcal{L} of length at most $\Gamma 2^{k'-n}$, $\text{Var}(\int_{\mathcal{L}} \Delta \tilde{\eta}_{N, k'}(z) |dz|) = 4^{k'-n} |\mathcal{L}|$. Here $|\mathcal{L}|$ is the length of \mathcal{L} . Furthermore if $v \in \mathbb{R}^2$ is orthogonal to \mathcal{L} , then

$$\text{Var}\left(\int_{\mathcal{L}} \Delta \tilde{\eta}_{N, k'}(z) |dz| - \int_{\mathcal{L}+v} \Delta \tilde{\eta}_{N, k'}(z) |dz|\right) = 4^{k'-n} \Theta(|\mathcal{L}|),$$

whenever $|\mathcal{L}| \geq 2^{k'-n}$ and $|v| = \Theta(1)$. Compare this to a similar estimate derived in the proof of Lemma 2.2.

We can now use strategies similar to those used for constructing cross_n . Since the fields $\tilde{\eta}_{N,k'}$'s do not have rotational invariance, we will actually construct crossings in all possible directions at any given scale and consider the *maximum expected weight* of these crossings. In view of properties (a) and (b), we can then obtain recursion relations like (4.3) and (4.12) on the maximum expected weight without any significant change in the analysis. Next we create a (lattice) crossing P_n^* of $\frac{1}{N}V_N^\Gamma$ from the crossing P_n which we constructed for V_N^Γ so that

$$\mathbb{E}\left(\sum_{v \in P_n^*} e^{\gamma \tilde{\eta}_{N,n}(v)}\right) = O_{\gamma,\varepsilon}(N^{1-\Omega(\gamma^{4/3}/\log \gamma^{-1})}).$$

We can do this by following the procedure detailed in the proof of Lemma 6.4. Indeed we have an analogous upper bound on $\text{Var}(\tilde{\eta}_{N,n}(v) - \tilde{\eta}_{N,n}(w))$ for adjacent v, w :

$$\max_{v,w \in V_N^\Gamma, |v-w|=1} \text{Var}(\tilde{\eta}_{N,n}(v) - \tilde{\eta}_{N,n}(w)) = 1,$$

which makes all the arguments employed in the proof of Lemma 6.4 work smoothly. The approximation of $\tilde{\eta}_{\gamma,N}$ with $\tilde{\eta}_{N,n}$ can be dealt with in a similar way as we did in Section 5. Once we have bound on expected weights of crossings between shorter boundaries of rectangles at all scales, we can use such crossings to build an efficient path connecting any two given points in V_N (we discussed this idea in Section 5 in greater detail). This leads to a proof of Theorem 1.3.

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